ISSN 0001-4346, Mathematical Notes, 2007, Vol. 82, No. 2, pp. 229–237. © Pleiades Publishing, Ltd., 2007.
Orininal English Text © A. M. Sedletskii, 2007, published in Matematicheskie Zametki, 2007, Vol. 82, No. 2, pp. 262–271.

Asymptotics of the Zeros of Degenerate Hypergeometric Functions

A. M. Sedletskii*

Moscow State University

Received November 15, 2005

Abstract—We find the asymptotics of the zeros of the degenerate hypergeometric function (the Kummer function) $\Phi(a, c; z)$ and indicate a method for numbering all of its zeros consistent with the asymptotics. This is done for the whole class of parameters a and c such that the set of zeros is infinite. As a corollary, we obtain the class of sine-type functions with unfamiliar asymptotics of their zeros. Also we prove a number of nonasymptotic properties of the zeros of the function Φ.

DOI: 10.1134/S0001434607070280

Key words: *degenerate hypergeometric function (Kummer function), asymptotics of zeros, sine-type function, Kummer's formula, Laplace transform, entire function.*

1. INTRODUCTION

We consider the *degenerate hypergeometric function* (the *Kummer function*)

$$ {}_1F_1(a, c; z) = \Phi(a, c; z) := 1 + \sum_{s=1}^{\infty} \frac{a(a+1)\cdots(a+s-1)}{c(c+1)\cdots(c+s-1)} \frac{z^s}{s!}, \qquad c \notin -\mathbb{Z}_+ \tag{1} $$

(see, for example, [1]–[3]), as a function of the variable z for fixed values of the parameters $a, c \in \mathbb{C}$. If $a \in -\mathbb{Z}_+$, then the series (1) terminates, i.e., Φ is a polynomial. Therefore, we assume the condition $a \notin -\mathbb{Z}_+$ equivalent to the assumption that Φ is an integer transcendental function.

The class of entire functions (1) is quite wide; under the appropriate values of the parameters a and c, many well-known special functions are expressed in terms of the function (1) (for example, the modified Bessel function for $c = 2a$; see [3]) or coincide with it (for example, $\Phi(1, c; z) = \Gamma(c)E_1(z, c)$, where $E_\rho(z, \mu)$ is a function of Mittag–Leffler type; see [4]).

In the present paper, we solve the problem of finding the asymptotics of the zeros of the function $\Phi(a, c; z)$ and of numbering all of its zeros consistently with the asymptotics. The solution of this problem has been given in a few special cases, such as the ones cited above (see, respectively, [5] and [6], [7]). However, apparently, this problem has not been investigated for the whole class of admissible parameters a and c such that the set of zeros of the function (1) is infinite.

As a corollary, we obtain the class of sine-type functions with (apparently) new asymptotics of zeros.

Besides, we prove a theorem on the nonasymptotic properties of the zeros of the function (1), namely, on their distribution in the right or left half-planes as well as in the horizontal strips

$$ (2n-1)\pi < |\operatorname{Im} z| < 2\pi n, \qquad n \in \mathbb{N} $$

(both for real parameters a and c from prescribed sets).

*E-mail: sedlet@mail.ru.

2. MAIN RESULT

Suppose that the sequence $Z = (z_n) \subset \mathbb{C}$ has the asymptotics

$$z_n = \varphi(n) + o(1), \qquad n \to \pm\infty, \tag{2}$$

where $\varphi(n) \to \infty$. The *numbering* of the sequence Z is said to be *consistent with the asymptotics* (2) via the index set T if there exists a bijection $T \leftrightarrow Z$ preserving this asymptotics.

Theorem 1. *Suppose that $a, c \in \mathbb{C}$ and $a, c, c-a \notin -\mathbb{Z}_+$. Then*

1) *all the zeros z_n of the function $\Phi(a, c; z)$ are simple and they satisfy the asymptotic formula*

$$\begin{aligned} z_n = 2\pi i n &+ \left((c-2a)\log 2\pi|n| + \log\frac{\Gamma(a)}{\Gamma(c-a)} \pm i\frac{\pi}{2}(c-2)\right)\left(1 + \frac{c-2a}{2\pi i n}\right) \\ &+ \frac{2a(a-c)-c}{2\pi i n} + O\left(\frac{\log|n|}{n^2}\right), \qquad n \to \pm\infty; \end{aligned} \tag{3}$$

2) *the numbering of all the zeros of the function $\Phi(a, c; z)$ is consistent with the asymptotic expansion* (3) *via the index set $T = \mathbb{Z} \setminus \{0\}$.*

Proof. 1) Since the function $w = \Phi(a, c; z)$ is a solution of the equation [3, Chap. 7, Sec. 9]

$$zw'' + (c-z)w' - aw = 0$$

for all $z \in \mathbb{C}$, the absence of multiple zeros of the function is proved in the same way as for the Bessel function [3, Chap. 7, Sec. 6], i.e., by the uniqueness of the solution of the Cauchy problem. To avoid repetitions, we omit the details.

To obtain formula (3), we start from the well-known asymptotic expansion (see [1], [2])

$$\begin{aligned} \Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}&\left(1 - \frac{a(1+a-c)}{z} + O\left(\frac{1}{z^2}\right)\right) \\ + \frac{\Gamma(c)}{\Gamma(a)} z^{a-c}e^z&\left(1 + \frac{(1-a)(c-a)}{z} + O\left(\frac{1}{z^2}\right)\right), \qquad z \to \infty, \end{aligned} \tag{4}$$

$$-\pi < \arg z \le \pi, \qquad -\pi < \arg(-z) \le \pi, \tag{5}$$

For brevity, we adopt the notation

$$\gamma = \frac{\Gamma(a)}{\Gamma(c-a)}, \qquad A = 2a(a-c) - c.$$

If $a, c-a \notin -\mathbb{Z}_+$, then γ is meaningful and $\gamma \ne 0$. Since

$$\left(1 - \frac{a(a-c+1)}{z} + O\left(\frac{1}{z^2}\right)\right) \Big/ \left(1 + \frac{(1-a)(c-a)}{z} + O\left(\frac{1}{z^2}\right)\right) = 1 + \frac{A}{z} + O\left(\frac{1}{z^2}\right),$$

formula (4) yields the following equation for sufficiently large (in absolute value) zeros z_n of the function $\Phi(z) = \Phi(a, c; z)$:

$$z^{a-c}(-z)^a e^z = -\gamma\left(1 + \frac{A}{z} + O\left(\frac{1}{z^2}\right)\right). \tag{6}$$

This implies that such zeros are located on the set

$$\arg z \mp \frac{\pi}{2} = O\left(\frac{\log|z|}{|z|}\right), \qquad |z| \ge r_0.$$

If $0 < \arg z < \pi$ (respectively, $-\pi < \arg z < 0$), then, by condition (5), we have $-\pi < \arg(-z) < 0$ (respectively, $0 < \arg(-z) < \pi$). Therefore,

$$(-z)^a = z^a e^{\mp i\pi a}, \qquad \operatorname{Im} z \gtrless 0. \tag{7}$$

Taking this into account and setting $-1 = e^{\mp i\pi}$, respectively, in the cases $\operatorname{Im} z \gtrless 0$, we can write Eq. (6) in the form

$$e^{z+(2a-c)\log z} = \gamma e^{\mp i\pi(1-a)}\biggl(1+\frac{A}{z}+O\Bigl(\frac{1}{z^2}\Bigr)\biggr). \tag{8}$$

This implies that, for sufficiently large $|n|$,

$$z_n + (2a-c)\log z_n = 2\pi i n \mp i\pi(1-a) + \log\gamma + \frac{A}{z_n} + O\Bigl(\frac{1}{n^2}\Bigr).$$

Twice iterating this formula, we obtain (3). The assertion 1) it is proved.

2) *Step* 1. Assuming $r_0 > 0$ to be sufficiently large and denoting $K(r_0) = (z : |z| > r_0)$, we introduce the sets

$$P_+ = (z : |z^{a-c}(-z)^a e^z| > 2|\gamma|) \cap K(r_0),$$
$$P_- = \Bigl(z : |z^{a-c}(-z)^a e^z| < \frac{1}{2}|\gamma|\Bigr) \cap K(r_0),$$
$$P = \Bigl(z : \frac{1}{2}|\gamma| \le |z^{a-c}(-z)^a e^z| \le 2|\gamma|\Bigr) \cap K(r_0),$$

which are, respectively, the right and left curvilinear half-planes and the union of two curvilinear half-strips located in the upper and lower half-planes. Obviously,

$$\mathbb{C} \setminus (z : |z| \le r_0) = P_+ \cup P_- \cup P.$$

We need convenient estimates for $|\Phi(z)|$ on these sets. To obtain them, we pass to the function

$$\Phi_1(z) = \frac{\Gamma(a)}{\Gamma(c)}(-z)^a\Phi(a,c;z).$$

It follows from (4) that

$$\Phi_1(z) = z^{a-c}(-z)^a e^z\biggl(1+O\Bigl(\frac{1}{z}\Bigr)\biggr) + \gamma\biggl(1+O\Bigl(\frac{1}{z}\Bigr)\biggr), \qquad z \to \infty. \tag{9}$$

On the sets P_+, P_-, the estimates $|\Phi_1(z)|$ can be obtained immediately from (9), the triangle inequality, and the definitions of these sets. Indeed, for a sufficiently large r_0 from (9), we obtain the inequalities

$$|\Phi_1(z)| \le \frac{3}{2}|z^{a-c}(-z)^a e^z| + 2|\gamma|, \tag{10}$$
$$|\Phi_1(z)| \ge \frac{3}{4}|z^{a-c}(-z)^a e^z| - \frac{5}{4}|\gamma|, \tag{11}$$
$$|\Phi_1(z)| \ge -\frac{5}{4}|z^{a-c}(-z)^a e^z| + \frac{3}{4}|\gamma|. \tag{12}$$

Hence if $z \in P_+$, then inequalities (10), (11) imply the estimates

$$\frac{1}{8}|z^{a-c}(-z)^a e^z| \le |\Phi_1(z)| \le \frac{5}{2}|z^{a-c}(-z)^a e^z|, \qquad z \in P_+, \tag{13}$$

but if $z \in P_-$, then inequalities (10), (12) imply the estimates

$$\frac{1}{8}|\gamma| \le |\Phi_1(z)| \le \frac{11}{4}|\gamma|, \quad z \in P_-. \tag{14}$$

If $z \in P$, then (10) yields the upper bound

$$|\Phi_1(z)| \le 5|\gamma|, \qquad z \in P. \tag{15}$$

However, we cannot obtain a lower bound as easily as on the sets P_+, P_-, because the set P contains the zeros of the function $\Phi_1(z)$. Therefore, we use the mapping

$$w = z + (2a-c)\log z. \tag{16}$$

It follows from (7) that

$$\operatorname{Re} w = \operatorname{Re}(z + (2a - c)\log z) = \log|z^{a-c}(-z)^a e^z| \mp \pi \operatorname{Im} a,$$

respectively, for $\operatorname{Im} z \gtrless 0$; therefore, for sufficiently large $|\operatorname{Im} w|$, the images of the components of the set P will coincide with the half-strips

$$\log\frac{1}{2}|\gamma| \mp \pi \operatorname{Im} a \le |\operatorname{Re} w| \le \log 2|\gamma| \mp \pi \operatorname{Im} a, \qquad \operatorname{Im} w \gtrless 0, \quad |\operatorname{Im} w| > v_0. \tag{17}$$

By (9), the image $\psi(w)$ of the function $\Phi_1(z)$ under the mapping (16) is of the form

$$\psi(w) = e^w e^{\mp i\pi a}(1 + o(1)) + \gamma(1 + o(1)), \qquad \operatorname{Im} w \to \infty, \quad \operatorname{Im} w \gtrless 0.$$

Obviously, in the half-strips (17), but outside the circles of fixed radius δ centered at at the zeros of the function $\psi(w)$ (i.e., at the points $w_n = 2\pi i n + \log\gamma \mp i\pi(1 - a) + o(1)$, $n \to \pm\infty$) the following estimate holds:

$$|\psi(w)| \ge C(\delta) > 0.$$

Since δ can be taken arbitrarily small, this implies the existence of a sequence $r_k \uparrow +\infty$ such that

$$|\Phi_1(z)| \ge C_0 > 0, \qquad z \in P, \quad |z| = r_k. \tag{18}$$

Now, on the basis of (13)–(15) and (18), we can write the resulting estimates for the function $\Phi(z)$:

$$|\Phi(z)| \asymp e^x r^{\operatorname{Re}(a-c)}, \qquad z = x + iy = re^{i\theta} \in P_+, \tag{19}$$

$$|\Phi(z)| \asymp r^{-\operatorname{Re} a}, \qquad z \in P_-, \tag{20}$$

$$|\Phi(z)| \asymp r^{-\operatorname{Re} a}, \qquad z \in P, \quad r = r_k \uparrow +\infty. \tag{21}$$

Step 2. It follows from the definition of the set P_+ that, on its boundary, we have

$$x \asymp \log r \quad \text{for} \quad \operatorname{Re}(2a - c) \ne 0 \qquad \text{and} \qquad x = O(1) \quad \text{for} \quad \operatorname{Re}(2a - c) = 0$$

(for $r > r_0$), whence

$$\cos\theta = \frac{x}{r} = O\left(\frac{\log r}{r}\right), \qquad r \to \infty. \tag{22}$$

Hence $\cos\theta \to 0$, $r \to \infty$ and $\theta \to \pm\pi/2$, $r \to \infty$, respectively, for $\operatorname{Im} z \gtrless 0$. Denote by $r\exp(i\theta^\pm)$ the points of intersection of the boundary of the set P_+ with the circle $|z| = r > r_0$, $\theta^\pm \gtrless 0$. Then, for $\theta = \theta^\pm$, we have relation (22) and, applying the reduction formulas $\cos\theta^\pm = \sin(\pi/2 \mp \theta^\pm)$, we find

$$\theta^\pm = \pm\frac{\pi}{2} + O\left(\frac{\log r}{r}\right), \qquad r \to \infty. \tag{23}$$

Step 3. The concluding step in the proof is based on Jensen's formula

$$\int_0^r \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\Phi(re^{i\theta})|\,d\theta, \tag{24}$$

where $n(t)$ is the number of zeros of the function $\Phi(z) = \Phi(a, c; z)$ in the disk $|z| < t$. We write

$$\int_{-\pi}^{\pi} \log|\Phi(re^{i\theta})|\,d\theta = \int_{\theta^-}^{\theta^+} + \int_{\theta^+}^{\pi} + \int_{-\pi}^{\theta^-} =: J_1 + J_2 + J_3. \tag{25}$$

By estimates (19) and relations (23), we have

$$\begin{aligned} J_1 &= \int_{\theta^-}^{\theta^+} (r\cos\theta + \operatorname{Re}(a - c)\log r)\,d\theta + O(1) \\ &= r(\sin\theta^+ - \sin\theta^-) + (\theta^+ - \theta^-)\operatorname{Re}(a - c)\log r + O(1) \\ &= 2r + \pi\operatorname{Re}(a - c)\log r + O(1), \qquad r \to \infty. \end{aligned}$$

To estimate J_2, J_3, we use estimates (20), (21), and formula (23). We obtain

$$J_2 + J_3 = \left(\int_{\theta^+}^{\pi} + \int_{-\pi}^{\theta^-}\right)(-\operatorname{Re} a \log r + O(1))\, d\theta$$
$$= \left(\pi + O\left(\frac{\log r}{r}\right)\right)(-\operatorname{Re} a \log r + O(1)) = -\pi \operatorname{Re} a \log r + O(1),$$

where $r = r_k$ is the sequence from estimate (21). Substituting the obtained estimates for J_k into formula (25) and then the resulting relation into (24), we obtain the following asymptotics:

$$\int_0^r \frac{n(t)}{t}\, dt = \frac{r}{\pi} - \frac{\operatorname{Re} c}{2} \log r + O(1), \qquad r = r_k \to +\infty. \tag{26}$$

On the other hand, we can estimate the left-hand side of (26) using formula (3). By (3), there exists an integer m such that if Z is the sequence of all zeros of the function $\Phi(z)$, then

$$Z = Z_+ \cup Z_-, \qquad Z_+ = (z_n^+)_{n=m}^{+\infty}, \quad Z_- = (z_n^-)_{n=-1}^{-\infty}, \quad Z_+ \cap Z_- = \varnothing, \tag{27}$$

and relation (3) will hold for $z_n^\pm$ as $n \to \pm\infty$. Thus, the numbering (27) is consistent with the asymptotic expansion (3) and it remains to prove that $m = 1$.

It follows from (3) that

$$|z_n| = \left((\operatorname{Im} z_n)^2 + (\operatorname{Re} z_n)^2\right)^{1/2} = |\operatorname{Im} z_n| \left(1 + O\left(\left(\frac{\operatorname{Re} z_n}{\operatorname{Im} z_n}\right)^2\right)\right) = |\operatorname{Im} z_n| + o(1), \qquad n \to \pm\infty.$$

Combining this with formula (3), we obtain

$$|z_n^\pm| = 2\pi|n| \pm \operatorname{Im}(c - 2a) \log |n| + \pi\left(\frac{\operatorname{Re} c}{2} - 1\right)$$
$$\pm (\operatorname{Im}(c - 2a) \log 2\pi + \arg \gamma) + o(1), \qquad n \to \pm\infty. \tag{28}$$

Let us use the following lemma [7]: if the positive sequence $\Lambda = (\lambda_n)_{n=m}^{+\infty}$ is of the form

$$\lambda_n = a_1 n + a_2 \log n + a_3 + o(1), \quad a_1 > 0, \ a_2, a_3 \in \mathbb{R}, \qquad n \to +\infty,$$

and $\Lambda(t)$ is the number of points λ_n on $(0, t)$, then, as $r \to \infty$,

$$\int_0^r \frac{\Lambda(t)}{t}\, dt = \frac{r}{a_1} - \frac{a_2}{2a_1} \log^2 r + \left(\frac{1}{2} - m - \frac{a_3}{a_1} + \frac{a_2}{a_1} \log a_1\right) \log r + o(\log r). \tag{29}$$

We apply this lemma to the sequences $|Z_+| = (|z_n^+|)_{n=m}^{+\infty}$ and $|Z_-| = (|z_{-n}^-|)_{n=1}^{\infty}$ with the following parameters appearing in (28):

$$a_1 = 2\pi, \qquad a_2 = \pm \operatorname{Im}(c - 2a), \qquad a_3 = \pi\left(\frac{\operatorname{Re} c}{2} - 1\right) \pm (\operatorname{Im}(c - 2a) \log 2\pi + \arg \gamma). \tag{30}$$

In other words, relation (29) is first written for $\Lambda = |Z_+|$ and then for $\Lambda = |Z_-|$; moreover, in the latter case, we must set $m = 1$ because of the numbering (27). After that, we add the resulting relations term-by-term. On the right-hand side, the terms corresponding to the summands in (30) with different signs cancel out and we obtain

$$\int_0^r \frac{n(t)}{t}\, dt = \frac{r}{\pi} + \left(1 - m - \frac{\operatorname{Re} c}{2}\right) \log r + o(\log r), \qquad r \to \infty.$$

Comparing this with formula (26), we see that $m = 1$. Theorem 1 is proved. □

Remark 1. The choice of the value of $\log \gamma$ ($\gamma = \Gamma(a)/\Gamma(c - a)$) in formula (3) is not important.

Indeed, the substitution of one value of $\log \gamma$ by another one will only induce the renumbering of the sequence of zeros via the same index set, after which the asymptotic expansion (3) is preserved.

Remark 2. The assumptions of Theorem 1 concerning a and c specify the whole set of parameters for which the function (1) has infinitely many zeros.

Indeed, by Kummer's formula (see [3]), we have

$$e^{-z}\Phi(a,c;z) = \Phi(c-a,c,-z) \tag{31}$$

and if $a-c \in \mathbb{Z}_+$, then the right-hand side is a polynomial, which follows from Definition (1).

3. SINE-TYPE FUNCTIONS AND THEIR GENERALIZATIONS

Denote by S_α, $\alpha \in \mathbb{R}$, the class of entire functions of exponential type for which the following estimate holds:

$$|F(z)| \asymp |z|^{-\alpha} e^{\pi|\operatorname{Im} z|}, \qquad |\operatorname{Im} z| \geq h = h(F) > 0. \tag{32}$$

The class S_0 consists of so-called *sine-type functions* introduced in calculus by Levin [8]. The classes S_α and, in particular, the class S_0 play an important role in nonharmonic analysis (see [9]).

The function

$$F(z) = \int_{-\pi}^{\pi} e^{izt}\, d\sigma(t), \qquad \operatorname{var}\sigma(t) < +\infty, \tag{33}$$

is a sine-type function if and only if $\sigma(t)$ has jumps at two points $\pm\pi$ [9]; in this case the zeros z_n of the function $F(z)$ satisfy the condition

$$z_n = n + O(1), \qquad n \to \pm\infty. \tag{34}$$

A number of papers dealt with sine-type functions which were not Fourier–Stieltjes transforms i.e., not expressible as (33) (for more on this, see [9]); in all these cases, the zeros of the functions in question also satisfied condition (34).

It is well known [9] that an entire function of the form

$$F(z) = \int_{-\pi}^{\pi} e^{izt} \frac{k(t)\,dt}{(\pi^2 - t^2)^{1-a}}, \qquad 0 < \operatorname{Re} a < 1, \quad \operatorname{var} k(t) < +\infty, \;\; k(\pm\pi \mp 0) \neq 0, \tag{35}$$

belongs to the class $S_{\operatorname{Re} a}$ and its zeros are also of the form (34).

It turns out that, under certain conditions on a and c, the function

$$F(z) = e^{-i\pi z}\Phi(a,c;2\pi i z) \tag{36}$$

belongs to the class $S_{\operatorname{Re} a}$ and possesses a number of additional properties. The following statement is valid.

Theorem 2. *Suppose that $a, c \in \mathbb{C}$ and $a, c, c-a \notin -\mathbb{Z}_+$, where*

$$\operatorname{Re} c = 2\operatorname{Re} a. \tag{37}$$

Then

1) *the function* (36) *belongs to the class $S_{\operatorname{Re} a}$;*

2) *The following asymptotics for the zeros z_n of the function* (36) *is valid:*

$$\begin{aligned} z_n = n &+ \left(\frac{\operatorname{Im}(c-2a)}{2\pi}\log 2\pi|n| - \frac{i}{2\pi}\log\frac{\Gamma(a)}{\Gamma(c-a)} \pm \frac{c-2}{4}\right)\left(1 + \frac{\operatorname{Im}(c-2a)}{2\pi n}\right) \\ &- \frac{2a(c-a)-c}{4\pi^2 n} + O\left(\frac{\log|n|}{n^2}\right), \qquad n \to \pm\infty; \end{aligned} \tag{38}$$

3) *the numbering of all the zeros of the function* (36) *is consistent with the asymptotic expansion* (38) *via the index set $T = \mathbb{Z} \setminus \{0\}$;*

4) *if, besides,*

$$c \neq 2a, \tag{39}$$

then, for $\operatorname{Re} a = 0$, *the function* (36) *is not a Fourier–Stieltjes transform, and whenever* $0 < \operatorname{Re} a < 1$ *it cannot be expressed in the form* (35).

Proof. First, it follows from formula (4) that the function (36) is of exponential type. Second, this formula, together with condition (37), shows that the following estimates hold:

$$|\Phi(z)| \asymp |z|^{-\operatorname{Re} a}, \quad \operatorname{Re} z \le -h; \qquad |\Phi(z)| \asymp |z|^{-\operatorname{Re} a} e^{\operatorname{Re} z}, \quad \operatorname{Re} z \ge h > 0,$$

if h is sufficiently large. Hence, the function (36) satisfies estimate (32) with $\alpha = \operatorname{Re} a$, i.e., the function (36) belongs to the class $S_{\operatorname{Re} a}$, and hence assertion 1) is proved.

Assertions 2) and 3) immediately follow from Theorem 1, where we must now put $2\pi i z_n$ on the left-hand side of (3).

Finally, if condition (39) holds, then formula (38), which implies $z_n - n \sim C \log|n|$, $n \to \pm\infty$, $C \neq 0$, is incompatible with condition (34), which is necessary for representing the function $F(z)$ in the form (33) for $\operatorname{Re} a = 0$ and in the form (35) for $0 < \operatorname{Re} a < 1$. Assertion 4) is also valid. Theorem 2 is proved. □

Theorem 2 gives (apparently, for the first time) examples of sine-type function whose zeros possess asymptotics not obeying formula (34).

4. NONASYMPTOTIC PROPERTIES OF ZEROS

Theorem 3. 1) *Suppose that* $1 \le a < c \le a+1$ *and* $c \neq 2$ *if* $a = 1$. *Then all the zeros of the function* (1) *lie in the half-plane*

$$\operatorname{Re} z < -(\sqrt{a-1} + \sqrt{1-(c-a)})^2. \tag{40}$$

2) *Suppose that* $0 < a \le 1$, $c \ge 1 + a$; *moreover,* $c \neq 2$ *if* $a = 1$. *Then all the zeros of the function* (1) *lie in the half-plane*

$$\operatorname{Re} z > (\sqrt{c-a-1} + \sqrt{1-a})^2.$$

3) *Suppose that* $0 < a \le 1$, $a < c \le 1 + a$; *moreover,* $c \neq 2$ *if* $a = 1$. *Then all the zeros of the function* (1) *lie in the horizontal strips*

$$(2n-1)\pi < |\operatorname{Im} z| < 2\pi n, \qquad n \in \mathbb{N}. \tag{41}$$

Proof. The following formula holds:

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1}\, dt, \qquad \operatorname{Re} a,\ \operatorname{Re}(c-a) > 0 \tag{42}$$

(see [1]–[3]), which allows us to apply Pólya's theorem [10] and the author's theorem [11] on the zeros of the Laplace transform

$$F(z) = \int_0^1 e^{zt} f(t)\, dt, \qquad f \in L^1(0,1). \tag{43}$$

Suppose that the function $f(t) \not\equiv C$ is positive and differentiable in $(0,1)$ and

$$\frac{f'(t)}{f(t)} \ge -h, \qquad 0 < t < 1.$$

Then, by Pólya's theorem [10], all the zeros of the function (43) lie in the half-plane $\operatorname{Re} z < h$. Hence, in view of formula (42), we must find the minimum of the function $f'(t)/f(t)$ on $(0,1)$, where

$$f(t) = t^{\alpha}(1-t)^{\beta}, \qquad \alpha = a - 1 > -1, \quad \beta = c - a - 1 > -1. \tag{44}$$

We have excluded the case $\alpha = \beta = 0$ in which, by (42), all the zeros of the function $\Phi(z)$ are located on the boundary of the half-plane (40). We have

$$\frac{f'(t)}{f(t)} = \frac{\alpha - (\alpha+\beta)t}{t(1-t)}. \tag{45}$$

For the function (45) to be bounded below on $(0,1)$, it is necessary that the numerator be nonnegative for $t = 0, 1$; this imposes the condition $\beta \le 0 \le \alpha$. Thus (see (44)), we consider the values

$$-1 < \beta \le 0 \le \alpha. \tag{46}$$

If $\alpha + \beta = 0$ (i.e., $c = 2$), then $\alpha > 0$ and the minimum value of the function (45) on $(0,1)$ is equal to $4\alpha = 4(a-1)$. By Pólya's theorem, all the zeros of the function $\Phi(z)$ lie in the half-plane $\operatorname{Re} z < 4(1-a)$, which coincides with (40) for $c = 2$. If $\alpha = 0$ (i.e., $a = 1$) and $\beta < 0$, then the minimum value of the function (45) on $[0,1)$ is equal to $-\beta$. By Pólya's theorem, all the zeros of the function $\Phi(z)$ lie in the half-plane $\operatorname{Re} z < \beta$, which, for $a = 1$, coincides with (40). Suppose that $\alpha + \beta \ne 0$, $\alpha > 0$. Then

$$\left(\frac{f'(t)}{f(t)}\right)' = -\frac{\alpha}{t^2} - \frac{\beta}{(1-t)^2}. \tag{47}$$

The middle part of this formula vanishes at the points

$$t_1 = \frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{-\beta}}, \qquad t_1 = \frac{\sqrt{\alpha}}{\sqrt{\alpha} - \sqrt{-\beta}}.$$

It is obvious that $t_1 \in (0,1]$, $t_2 \ge 1$ and the minimum value of the function (45) on $(0,1]$ is attained at the point t_1. Substituting this value into the right-hand side of (45), we obtain

$$\frac{f'(t)}{f(t)} \ge (\sqrt{\alpha} + \sqrt{-\beta})^2, \qquad 0 < t < 1.$$

Combining this result with Pólya's theorem, we obtain assertion 1). Here we must take into account the fact that condition (46) passes into the condition $1 \le a < c \le a + 1$.

Assertion 2) follows from assertion 1) and formula (31).

To prove assertion 3), we use the following result [11]: if the function $f(t) \not\equiv C$ is positive and logarithmically convex in $(0,1)$, then all the zeros of the function (43) lie in the strips (41). Besides, it remains to find the set of parameters $\alpha, \beta > -1$, for which the function (44) is logarithmically convex in $(0,1)$, i.e., the function $\log f(t)$ is convex in $(0,1)$. Since $(\log f(t))''$ is the right-hand side of (47), the function $\log f(t)$ is convex in $(0,1)$ only for values of $\alpha, \beta > -1$ such that

$$\alpha(1-t)^2 + \beta t^2 \le 0 \tag{48}$$

for $t \in (0,1)$ and, by continuity, also for $t \in [0,1]$. Substituting the values $t = 0, 1$ into inequality (48), we obtain the necessary condition $\alpha, \beta \le 0$. Obviously, it is also sufficient for (48) to hold. It remains to recall the relation (44) between the pairs of parameters α, β and a, b. Assertion 3) is proved. The proof of Theorem 3 is complete. □

In connection with Theorem 3, note the papers of Tsvetkov [12], [13] on the zeros of the Whittaker function

$$M_{k,m}(z) = e^{-z/2} z^{m+1/2} \Phi\left(\frac{1}{2} + m - k, 1 + 2m; z\right). \tag{49}$$

On the basis of formula (49), we shall now restate some of the results from [12], [13] for the function $\Phi(a, c; z)$.

If $0 < c < 2a$ (respectively, $c > \max(0, 2a)$), it follows that the function (1) has no complex zeros in the half-plane $\operatorname{Re} z \ge c - 2a$ (respectively, $\operatorname{Re} z \le c - 2a$) [12].

If $1 < c < 2a$ (respectively, $c > \max(1, 2a)$), then the function (1) has complex zeros only in the half-plane $\operatorname{Re} z < c - 2a$ (respectively, $\operatorname{Re} z > c - 2a$) [13].

We see that the result from [12] includes a wider set of parameters a and c as compared to Theorem 3. On the other hand, Theorem 3 indicates the half-plane of zeros more exactly. Indeed, if the inequalities $1 < a < c < a + 1$ hold, then

$$-(\sqrt{a-1} + \sqrt{1-(c-a)})^2 < c - 2a;$$

therefore, the half-plane (40) from Theorem 3 is a proper subset of the half-plane $\operatorname{Re} z < c - 2a$ appearing in [12][13].

ACKNOWLEDGMENTS

The author wishes to express gratitude to M. K. Kerimov and A. A. Muromskii for the possibility to use the rare edition [1].

This work was supported by the Russian Foundation for Basic Research (grant no. 06-01-00326) and the program "Leading Scientific Schools" (grant no. 4564.2006.1).

REFERENCES

1. L.-J. Slater,, *Confluent Hypergeometric Functions* (Cambridge University Press, New York, 1960; Vychisl. Tsentr AN SSSR, Moscow, 1966) [in Russian].
2. E. Jahnke, F. Emde, and F. Lösch, *Tafeln Höherer Funktionen* (Teubner, Stuttgart, 1960; Nauka, Moscow, 1977).
3. F. W. J. Olver, *Asymptotics and Special Functions* (Academic Press, New York–London, 1974; Nauka, Moscow, 1978).
4. M. M. Dzhrbashyan, *Integral Transformations and Representation of Functions in the Complex Domain* (Nauka, Moscow, 1966) [in Russian].
5. G. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, Cambridge, 1945; Inostr. Lit., Moscow, 1949), Vol. 1.
6. A. M. Sedletskii, "Asymptotic formulas for zeros of a function of Mittag–Leffler type," Anal. Math. **20** (2), 117–132 (1994).
7. A. M. Sedletskii, "On zeros of functions of Mittag–Leffler type," Mat. Zametki [Math. Notes] **68** (5), 710–724 (2000) [**68** (5–6), 602–613 (2000)].
8. B. Ya. Levin, "Interpolation by entire functions of exponential type," in *Mathematical Physics and Functional Analysis* (FTINT AN Ukr. SSR, 1969), Vol. 1, pp. 136–146 [in Russian].
9. A. M. Sedletskii, *Classes of Analytic Fourier Transforms and Exponential Approximations* (Fizmatlit, Moscow, 2005) [in Russian].
10. G. Polya, "Über die Nullstellen gewisser ganzer Funktionen," Math. Z. **2** (3–4), 352–383 (1918).
11. A. M. Sedletskii, "On the zeros of Laplace transforms," Mat. Zametki [Math. Notes] **76** (6), 883–892 (2004) [**76** (6), 824–833 (2004)].
12. G. E. Tsvetkov, "On the roots of Whittaker functions," Dokl. Akad. Nauk SSSR **32** (1), 10–12 (1941).
13. G. E. Tsvetkov, "On the complex roots of the Whittaker function," Dokl. Akad. Nauk SSSR **33** (4), 290–291 (1941).