A Bessel polynomial argument to prove the Riemann Hypothesis

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Abstract

The Gauss-Weierstrass density function

$$f(x) := e^{-\pi x^2}$$

enables a representation of Riemann's duality equation in the form

$$\xi(s) = \zeta(s)(1-s)\left[\int_0^\infty x^s f'(x)dx\right] = \zeta(1-s)s\left[\int_0^\infty x^{1-s} f'(x)dx\right] .$$

The modified Bessel-Hankel functions $\Psi_\nu(x) := \arctan\left[Y_\nu(x)/J_\nu(x)\right]$ and

$$\varphi_{2\nu}(x) := \frac{\pi}{2}\left[J_\nu^2(x) + Y_\nu^2(x)\right] = \frac{2}{\pi} 2\cos\pi\nu \int_0^\infty\int_0^\infty e^{-2x\sinh t\cosh\tau} \cosh(2\nu\tau)d\tau dt$$

are linked by the relation $x\varphi_{2\nu}(x)d\Psi_\nu = dx$ with a $\log(x)-$singularity at zero ([27] G. N. Watson). For an appropriately defined $\Gamma_{2\nu}(s/2)$ and $\mathrm{Re}(s \pm 2\nu) > 0$ it holds

$$\int_0^\infty x^s \varphi_{2\nu}(x)\frac{dx}{x} = \int_0^\infty x^{1/2-s}\left[\frac{1}{\sqrt{x}}\varphi_{2\nu}(x)\varphi_{2\nu}(\frac{1}{x})\right]d\Psi_{2\nu} = \Gamma_{2\nu}(\frac{s}{2}) .$$

As $\varphi_{2\nu}(x)$, which is an integral of a square of a real-valued function, has a representation in the form

$$\frac{1}{x}\varphi_{2\nu}(\frac{1}{x}) = \sum_0^\infty P_{n.2\nu}x^{2n} \quad , \quad P_{n.2\nu} := \frac{1*3*...(2n-1)}{2^n}(\nu, n)$$

and as $x\varphi_{2\nu}(x)$ defines a distribution function for $0 < \mathrm{Re}(2\nu) < 1$ the corresponding density function $\frac{d}{dx}\left[x\varphi_{2\nu}(x)\right]$ might leverage existing techniques or conjectures to prove the RH, e.g. Ramanujan's Master Theorem, Polya's argument about the zeros of certain entire functions or the Berry conjecture, when replacing the Gauss-Weierstrass density by the Bessel density function, taking advantage of an appropriate choice of the parameter 2ν .

Every feedback to this paper is highly appreciated: fuchs.braun@gmx.de

0. Introduction and Preliminaries

Our terminology follows those of [8] H.M. Edwards and [27] G. N. Watson. Throughout this paper we denote with $\log x$ the natural logarithm, i.e. $\log x = \log_e x = \ln x$.

The Gauss-Weierstrass density function

(0.1) $$f(x) := e^{-\pi x^2}$$

fulfills the following key properties

i) $\int_0^\infty x^s f(x) \frac{dx}{x}$ converge, defining the Gamma function e.g. in the form

(0.2) $$\Pi^*(s) := \Gamma(1+\frac{s}{2})\pi^{-s/2} = \int_0^\infty x^s \frac{d}{dx}\left[e^{-\pi x^2}\right] dx$$

ii) $xf(x) \approx f'(x)$ is a decreasing function for $x \to \infty$

iii) the Fourier transform relation $\left[\frac{1}{\sqrt{2\pi\sigma}} e^{-\sigma y^2}\right]^\wedge = \frac{1}{\sigma\sqrt{4\pi}} e^{-\frac{\xi^2}{4\sigma}}$.

For the entire function

(0.3) $$\xi(s) := \Pi^*(s)(s-1)\zeta(s) = (s-1)\zeta(s)\Psi(\frac{s}{2}) = \frac{s}{2}(s-1)\int_0^\infty x^{s/2}\psi(x)\frac{dx}{x}$$

(0.4) with $$\psi(x^2) := \sum_1^\infty e^{-\pi n^2 x^2}$$

Riemann's duality equation, valid for all complex values s,

(0.5) $$\xi(s) := \zeta(s)(s-1)\Pi^*(s) = \xi(1-s)$$

can be written in the form

(0.6) $$\xi(s) = \zeta(s)(1-s)\left[\int_0^\infty x^s f'(x)dx\right] = \zeta(1-s)s\left[\int_0^\infty x^{1-s} f'(x)dx\right] .$$

applying Jacobi's ϑ – relation ([8] H.M. Edwards 10.3)

(0.7) $$G(x) := \sum_{-\infty}^{+\infty} e^{-\pi n^2 x^2} = \sum_{-\infty}^{\infty} f(nx) = 1 + 2\sum_1^{+\infty} e^{-\pi n^2 x} =: 1 + 2\psi(x^2) = G(\frac{1}{x}) .$$

An equivalent formula to (0.7) is given in [12] H. Hamburger, by

(0.8) $$i\cot i\pi x = 1 + 2\sum_1^\infty e^{-2\pi n x} = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_1^\infty \frac{1}{x^2+n^2}$$

The hyperbolic secant function

(0.9) $$g_a(t) := \frac{1}{\cosh at}$$

fulfills similar properties than (0.1), e.g. for its Fourier transform it holds

(0.10) $$\hat{g}_a(\omega) := \frac{\pi}{a}\frac{1}{\cosh \pi^2 \frac{\omega}{a}} = \frac{\pi}{a} g(\pi^2 \frac{\omega}{a})$$

and ([17] B. E. Petersen, chapter 2, §3)

(0.11) $$\frac{\pi}{4\cosh \pi t} = \sum_1^\infty (-1)^{k+1} \frac{\frac{2k-1}{2}}{\left[\frac{2k-1}{2}\right]^2 + t^2} = \sum_1^\infty \left[(-1)^{k+1} e^{-\frac{2k-1}{2}t} \right]^\wedge$$

From [15] N. Nielsen, chapter IV, §22, 23, 25 we recall

(0.12) $$\left|\Gamma(1/2+it)\right|^2 = \frac{\pi}{\cosh \pi t} = \left|\Gamma(1/2-it)\right|^2 \quad , \quad t\left|\Gamma(it)\right|^2 = \frac{\pi}{\sinh t\pi} \quad .$$

From [11] I.S. Gradshteyn, I.M. Ryzhik, 9.61 9.63, [27] G.N. Watson, 7-51) we note

(0.13) $$\frac{t}{e^t - 1} = \sum_0^\infty B_{2n} \frac{t^{2n}}{(2n)!} \quad , \quad \frac{t}{2}\cot\frac{t}{2} = 1 - \sum_1^\infty B_{2n} \frac{t^{2n}}{(2n)!}$$

$$\frac{1}{\cosh t} = \sum_0^\infty E_{2n} \frac{t^{2n}}{(2n)!} \quad , \quad \frac{1}{\cos t} = \sum_0^\infty (-1)^n E_{2n} \frac{t^{2n}}{(2n)!}$$

with Bernoulli numbers B_n and Euler numbers E_n .

The corresponding "ϑ – relation" of (0.9) to (0.7) is given by

(0.14) $$G^*(x) := \sum_{n\in Z} \frac{1}{\cosh(\frac{nx}{2})} = \sum_{n\in Z} \left[\frac{1}{\cosh \pi n x} \right]^\wedge = \frac{1}{x} \sum_{n\in Z} \frac{1}{\cosh \frac{\pi n}{x}} = \frac{1}{x} G^*(\frac{2\pi}{x}) \ ,$$

whereby

(0.15) $$\int_0^\infty \frac{x^s}{\cosh x}\frac{dx}{x} = \Gamma(s)\zeta^*(s) \quad \text{and} \quad \int_0^\infty \frac{x^s}{\sinh x}\frac{dx}{x} = \frac{2^s - 1}{2^{s-1}}\Gamma(s)\zeta(s) \quad \text{(i.e. formally } \zeta(1) = \int_0^\infty \frac{dt}{\sinh t} \text{).}$$

with

(0.16) $$\zeta^*(s) := \sum_1^\infty (-1)^{k+1} \frac{2}{(2k-1)^s} \quad \text{and} \quad \zeta^*(2n+1) = \frac{\pi^{2n}}{2^{2n+2}(2n)!}\left|E_{2n}\right| \quad ,$$

The related formulas of the Zeta function are

$$\zeta(s) := \sum_1^\infty \frac{1}{k^s} \quad \text{and} \quad \zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad .$$

The corresponding proof of (0.5) for (0.14) leads to a representation in the form

(0.18) $$2^{1+s}\Gamma(s)\zeta(s)\zeta^{*}(s)=\int_{1}^{\infty}x^{s}\psi^{*}(x)\frac{dx}{x}+2\pi\int_{1}^{\infty}x^{1-s}\psi^{*}(2\pi x)\frac{dx}{x}+\frac{(2\pi)^{s}}{2s(1-s)}\quad .$$

Our Bessel polynomial approach uses alternatively to (0.1) the Bessel functions of 1st and 2nd kind

(0.19) $$\varphi_{2\nu}(x):=\frac{\pi}{2}\left[J_{\nu}^{2}(x)+Y_{\nu}^{2}(x)\right]=\frac{\pi}{2}\left|H_{\nu}^{(1)}(x)\right|^{2}=\frac{2}{\pi}\left|K_{\nu}(xe^{\pm\frac{\pi i}{2}})\right|^{2}$$

$$\varphi_{2\nu}(x)=\frac{\cos\pi\nu}{2\pi}\int_{0}^{\infty}\int_{0}^{\infty}e^{-2x\sinh t\cosh\tau}\cosh(2\nu\tau)d\tau dt=\frac{4}{\pi}\int_{0}^{\infty}K_{0}(2x\sinh t)\cosh(2\nu t)dt$$

(see appendix and [26] G.N. Watson 13-75). With the notation from [26] G.N. Watson 7-35)

$$\tan(\psi(x)):=\frac{Q(x,0)}{P(x,0)}\approx\frac{1}{8x}-\frac{33}{512x^{3}}+\frac{3147}{16384x^{5}}\pm$$

and

$$\arctan\left[\frac{Y_{\nu}(x)}{J_{\nu}(x)}\right]=\frac{\pi}{4}(2\nu-1)-\psi(x)$$

we recall from [26] G.N. Watson 15-52, 15-53, the following alternative representation

$$x\varphi_{2\nu}(x)=\frac{1}{1-\psi'(x)}\qquad .$$

The Mellin transform of $K_{\rho}(x)$, given in [26] G.N. Watson 13-21, we state in

Lemma 0.1: For $\operatorname{Re}(\mu)>\left|\operatorname{Re}(\rho)\right|$ it holds

(0.20) i) $$\int_{0}^{\infty}t^{\mu-1}K_{\rho}(t)dt=2^{\mu-2}\Gamma\left(\frac{\mu-\rho}{2}\right)\Gamma\left(\frac{\mu+\rho}{2}\right)$$

ii) $$\int_{0}^{\infty}y^{2n}K_{1}(y)\frac{dy}{y}=2^{2n-2}\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\qquad \text{for } n\in N$$

iii) $$\frac{2}{\pi}\int_{0}^{\infty}y^{k}K_{k}\left(\frac{y}{2}\right)\frac{dy}{y}=1*3*5....(2n-1)=\frac{(2n)!}{n!}\quad .$$

Lemma 0.1 is used to show

Lemma 0.2: For $\varphi_{2\nu}(x)$ and $\mathrm{Re}(x)>0$ the following asymptotic expansion holds true

(0.21) $$\frac{1}{x}\varphi_{2\nu}(\frac{1}{x})=\sum_0^\infty P_{n.2\nu}x^{2n}$$

with $$P_{n.2\nu}:=\frac{1}{2^{2n}}\frac{(2n)!}{n!}\frac{\Gamma(\nu+n+1/2)}{\Gamma(\nu-n+1/2)}\ .$$

We recall the proof of lemma 0.2 from [26] G.N. Watson 7-51, 13-75 in the appendix, to give some insight into the analysis technique. Some formulas to be used in this proof we recall in

Lemma 0.3 For $n\in N$ it holds

$$\frac{(2n)!}{2^{2n}n!}=\frac{\Gamma(1/2+n)}{\sqrt{\pi}}=\frac{(-1)^n}{\sqrt{\pi}\Gamma(1/2-n)} \quad \text{i.e.} \quad \Gamma^2(\frac{1}{2}+n)\Gamma^2(\frac{1}{2}-n)=\pi^2$$

and Stirling formula in the form

$$\lim\frac{\sqrt{\pi n}}{n!}\frac{(2n)!}{2^{2n}n!}=\lim\frac{\sqrt{n}}{n!}\Gamma(\frac{1}{2}+n)=\lim\frac{\sqrt{n}}{n!}\frac{(-1)^n}{\Gamma(1/2-n)}=1 \quad , \quad n\to\infty$$

Proof of lemma 0.3 is given in the appendix.

Substituting the variable $y=nx$ in lemma 0.3 leads to

Corollary 0.4 For $\mathrm{K}(x):=\sum_{n=1}^\infty K_1(2nx)$, $k\in N$ and $\mathrm{Re}(s)>1$ it holds

i) $$(\int_0^\infty y^{2k}K_1(y)\frac{dy}{y})(\int_0^\infty y^{2k}K_1(y)\frac{dy}{y})=2^{4k-4}\Gamma^2(k-\frac{1}{2})\Gamma^2(k+\frac{1}{2})=2^{4k-4}\pi^2 \quad \text{for } k\in N$$

ii) $$\zeta(4k)=(\frac{2}{\pi}\int_0^\infty x^{2k}\mathrm{K}(x)\frac{dx}{x})(\frac{2}{\pi}\int_0^\infty x^{2k}\mathrm{K}(x)\frac{dx}{x})=-\frac{(2\pi)^{4k}}{2(4k)!}B_{4k}\ .$$

From [26] G.N. Watson 15-61 we recall the identity

(0.22) $$\left|K_\nu(xe^{\pm\frac{\pi i}{2}})\right|^2=\left[\frac{\pi}{2}\right]^2\left[J_\nu^2(x)+Y_\nu^2(x)\right]$$

and note the special relation $$K_{1/2}(x)=\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}\ .$$

We use the abbreviation

(0.23) $$p_{n,2\nu} := ((2\nu)^2 - 1^2)((2\nu)^2 - 3^2)...((2\nu)^2 - (2n-1)^2)$$

$$(\nu, n) := \frac{\Gamma(\nu + n + 1/2)}{\Gamma(\nu - n + 1/2)}$$

and get from [26] G.N. Watson 7-22 and lemma 0.3

(0.24) $$(\nu, n) = \int^{(u+,1/u+,1+)} v^{n+\nu-1/2}(1-v)^{-2n-1} dv \ ,$$

with the contour integral $\int^{(u+,1/u+,1+)}$, given in [26] G.N. Watson 7-4, and

(0.25) $$(\nu, n) = \frac{p_{n,2\nu}}{2^{2n} n!} = \frac{(2n)!}{2^{2n} n!} \frac{p_{n,2\nu}}{(2n)!} = \frac{(-1)^n}{\sqrt{\pi}\Gamma(\frac{1}{2} - n)} \frac{p_{n,2\nu}}{(2n)!} \propto \frac{n!}{\sqrt{\pi n}} \frac{p_{n,2\nu}}{(2n)!} \quad .$$

The relation to the polynomials in (0.21) we summaries in

Lemma 0.5: For the polynomials in (0.21) it holds

i) $P_{n.2\nu} := \frac{1*3*...(2n-1)}{2^n}(\nu, n) \propto \frac{p_{n,2\nu}}{2^{2n}\sqrt{\pi n}}$

ii) $p_{n,0} = (-1)^n \left[1*3*...(2n-1)\right]^2$, $P_{n.0} := (-1)^n \frac{\left[1*3*...(2n-1)\right]^4}{(2n)! 2^{2n}}$

ii) $P_{n.2\nu} = \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2} + n)(\nu, n) = \frac{(-1)^n}{\pi} \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2} - n)} \frac{p_{n,2\nu}}{(2n)!}$

iv) $p_{-n,2\nu} = ((2\nu)^2 - (2n+1)^2) p_{n,2\nu}$

v) $F(n) := 2^{2n}(2n+1)^2 P_{n.0} = 2^{-2n}(2n+1)^{-2} P_{-n.0} = F(-n)$.

Proof of lemma 0.5 is given in the appendix.

The objectives of this paper are:

1. we propose the density function

$$\frac{d}{dx}\left[x\varphi_{2\nu}(x)\right]=\frac{d}{dx}\left[\sum_{0}^{\infty}P_{n.2\nu}x^{-2n}\right]=-\frac{1}{x}\sum_{1}^{\infty}2nP_{n.2\nu}\left[\frac{1}{x}\right]^{2n}$$

(lemma 1.1) as alternative to the Gauss-Weierstrass density function (0.1) to overcome current issues in the Zeta theory (e.g. [8] H.M. Edwards 10.3) .

2. we give the Mellin transform of (0.19), which is an integral of a square of a real-valued function, enabling e.g. the analysis technique from [10] G. Gasper, to prove that certain Mellin transforms of sums of squares have only real zeros.

3. we sketch a few options to prove the RH, based on an appropriate choice of the free parameter 2ν, using specific properties of the Bessel polynomials and the function $x\varphi_{2\nu}(x)$, given in lemma 1.1, e.g. that it defines a distribution function for $0<2\nu<1$.

In order to make a first link to Polya's arguments we recall

Lemma A (Polya): If φ is a polynomial which has all its roots on the imaginary axis, or if φ is an entire function which can be written in a suitable way as limit of such polynomials, then

If (*) $\int_0^\infty u^{1-s}F(u)\frac{du}{u}$ has all its zeros on the critical line, so does $\int_0^\infty u^{1-s}F(u)\varphi(\log u)\frac{du}{u}$.

In this context we also recall from [6] D. Bump et.al.

Remark B An operator which takes an even function $q(\nu)$ and replaces it by $\frac{q(\nu+1)-q(\nu-1)}{\nu}$ has the property of moving the zeros of a function closer on the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis.

1. Main result

This section gives the Mellin transform of the Bessel functions of 1st and 2nd kind

(1.1)
$$\varphi_{2\nu}(x) := \frac{\pi}{2}\left[J_\nu^2(x)+Y_\nu^2(x)\right] = \frac{2}{\pi}\left|K_\nu(xe^{\pm\frac{\pi i}{2}})\right|^2 .$$

We summaries the key properties of (1.1) in lemma 1.1. The proof resp. the references to the literature are given in the appendix.

We use the following abbreviations ([27] G.N. Watson 13-74)

$$\Theta_{2\nu}(x) := \tanh t - 2\nu\tanh 2\nu t > 0 \qquad \text{for } x>0 \text{ and } 0<2\nu<1$$

$$\Omega_{2\nu}(x) := \tanh t + 2\nu\tanh 2\nu t\sinh^2 t > 0 \quad \text{for } x \ge \nu \ .$$

Lemma 1.1 It holds

i) $$\varphi_{2\nu}(x) = \frac{4\cos\pi\nu}{\pi}\int_0^\infty\int_0^\infty e^{-2x\sinh t\cosh\tau}\cosh(2\nu\tau)d\tau dt = \frac{4\cos\pi\nu}{\pi}\sum_0^\infty a_{\nu,m}\int_0^\infty\int_0^\infty e^{-2x\sinh t\cosh\tau}\cosh\tau\sinh^{2m}\tau d\tau dt$$

for $-1<\mathrm{Re}(2\nu)<1$, $\mathrm{Re}(x)>0$,

ii) $$\varphi_{2\nu}(x)d\Psi_\nu = \frac{\pi}{2}\left[J_\nu^2(x)+Y_\nu^2(x)\right]d\Psi = \frac{dx}{x} \quad \text{with} \quad \Psi_\nu(x) := \arctan\left[\frac{Y_\nu(x)}{J_\nu(x)}\right],$$

iii) for $x>0$ and $0<2\nu<1$ the function $x\varphi_{2\nu}(x)$ is increasing with

$$0\uparrow x\varphi_{2\nu}(x)\uparrow 1 \qquad \text{for } 0\leftarrow x \text{ resp. } x\to\infty$$

due to

$$\frac{d}{dx}\left[x\varphi_{2\nu}(x)\right] = \frac{4}{\pi}\int_0^\infty K_0(2x\sinh t)\tanh t\cosh(2\nu t)\Theta_{2\nu}(t)dt > 0$$

iv) for $x\ge\nu$ the function $\sqrt{x^2-\nu^2}\,\varphi_{2\nu}(x)$ is increasing with

$$0\uparrow\sqrt{x^2-\nu^2}\,\varphi_{2\nu}(x)\uparrow 1 \qquad \text{for } \nu\leftarrow x \text{ resp. } x\to\infty$$

due to

$$\frac{d}{dx}\left[\sqrt{x^2-\nu^2}\,\varphi_{2\nu}(x)\right] = \frac{4}{\pi}\frac{x}{\sqrt{x^2-\nu^2}}\int_0^\infty K_0(2x\sinh t)\tanh t\cosh(2\nu t)\Omega_{2\nu}(x))dt > 0 \ .$$

We combine lemma 0.1 and lemma 0.2 in

Corollary 1.2 It holds

(1.3)
$$x\frac{d}{dx}\left[x\varphi_{2\nu}(x)\right] = -\sum_{1}^{\infty} 2nP_{n.2\nu}x^{-2n} > 0$$

In order to formulate the Mellin transform of (1.1) we use the notation from

Definition 1.3 For $\mathrm{Re}(s \pm 2\nu) > 0$ we put

(1.4a)
$$\Phi_{2\nu}(x) := \frac{1}{\sqrt{x}}\varphi_{2\nu}(x)\varphi_{2\nu}(\frac{1}{x}) = \frac{1}{x}\Phi_{2\nu}(\frac{1}{x})$$

(1.4b)
$$d\mu_{2\nu} := \sqrt{x}\Phi_{2\nu}(x)d\Psi_{2\nu} = \varphi_{2\nu}(\frac{1}{x})\frac{dx}{x}$$

(1.5)
$$\Gamma_{2\nu}(\frac{s}{2}) := \left[\frac{2}{\pi}\right]^{3/2} \cos\pi\nu \, \frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2})}{2^{\frac{1-s}{2}}2^{\frac{s}{2}}} \frac{\Gamma(\frac{s+2\nu}{2})\Gamma(\frac{s-2\nu}{2})}{2^{\frac{s+2\nu}{2}}2^{\frac{s-2\nu}{2}}})$$

Lemma 1.1 enables the calculation of the Mellin transform of (1.1), which we give in

Proposition 1.4 For $\mathrm{Re}(s \pm 2\nu) > 0$ it holds

i)
$$\int_0^\infty x^s\varphi_{2\nu}(x)\frac{dx}{x} = \int_0^\infty x^{-s}d\mu_{2\nu} = \int_0^\infty x^{1/2-s}\Phi_{2\nu}(x)d\Psi_{2\nu} = \Gamma_{2\nu}(\frac{s}{2}) \quad ,$$

ii)
$$\int_0^\infty x^s\varphi_0(x)\frac{dx}{x} = \frac{2}{\pi}B(\frac{1-s}{2},\frac{s}{2})\Gamma^2(\frac{s}{2}) \quad ,$$

iii)
$$\int_0^\infty x^s\varphi_{\pm 1/2}(x)\frac{dx}{x} = \frac{1}{2\pi}B(\frac{1-s}{2},\frac{s}{2})\Gamma(\frac{s+1/2}{2})\Gamma(\frac{s-1/2}{2}) \ \cdot$$

whereby

(1.6)
$$\frac{2^{s-1}}{\sqrt{2\pi}}\Gamma(\frac{s+1/2}{2})\Gamma(\frac{s-1/2}{2}) = \Gamma(s-\frac{1}{2}) \ \cdot$$

Proof of Proposition 1.4 is given in the appendix . •

Applying the same arguments for (1.3) leads to

Proposition 1.5 For $\mathrm{Re}(2\nu)$ and $\mathrm{Re}(....)>0$ it holds

$$\Gamma^*(s):=\int_0^\infty x^s\frac{d}{dx}\left[x\varphi_{2\nu}(x)\right]\frac{dx}{x}=\frac{8}{\pi}\Gamma^2(\frac{s}{2})\left[\sum_0^\infty a_{\frac{s}{2},m}\Gamma(m-\frac{s-3}{2})\Gamma((1-m)+\frac{s-3}{2})-\Gamma(1-\frac{s-3}{2})\Gamma(\frac{s-3}{2})\right] \quad .$$

Proof of Proposition 1.5 is given in the appendix . •

Applying the Müntz formula (see appendix) to

(1.7) $$\varphi_{2\nu}^*(x):=\sum_1^\infty\varphi_{2\nu}(nx)$$

gives

(1.8) $$\zeta(s)\int_0^\infty x^s\varphi_{2\nu}(x)\frac{dx}{x}=\int_0^\infty x^s\left[\varphi_{2\nu}^*(x)-\frac{1}{x}\int_0^\infty\varphi_{2\nu}(t)dt\right]\frac{dx}{x}$$ for $0<\mathrm{Re}(s)<1$.

We use (1.5) and (1.8) to sketch a few options to prove the RH on the next two pages:

From here on it's WIP (work in progress) only.

Option 1: If for $0<\mathrm{Re}(s)=:\sigma<1$ there is an appropriate setting of $2\nu:=\varphi(s)$ fulfilling a representation in the form

$$(*) \qquad \Gamma_{2\nu}(\frac{s}{2})=\chi(\varphi(s))\chi(\varphi(1-s)),$$

at least on the critical line, then there is a representation of the Zeta function as transform of a self-adjoint integral operator, which is positive definite at the same time. Therefore there is a underlying eigenfunctions/eigenvalues system, defining a corresponding Hilbert space, which gives the domain of the operator. This is the **Berry conjecture**.

The famous Gamma identity for $0<\mathrm{Re}(z)<1$

$$g(z)g(1-z)=\frac{\pi}{\sin\pi z} \quad \text{with} \quad g(z):=\Gamma(z) \ ,$$

might be seen as counterpart of (*): Its proof is using the "Haar measure" property on the multiplicative group of positive real numbers $(R^*_{>0},*)$) (see also [8] H.M. Edwards 10.2 for the corresponding Fourier analysis technique and handicaps in the context of self-adjoint operator and its transforms) of

$$d\mu(x):=\frac{dx}{x} \quad \text{with} \quad d\mu(cx)=\frac{d(cx)}{x}=d\mu(x)\cdot$$

to be applied to

$$\Gamma(z):=\int_0^\infty x^z e^{-x}d\mu(x) \quad .$$

It holds by replacing $x\to xy$ with $d\mu(xy)=d\mu(y)$

$$\Gamma(z)\Gamma(1-z)=\int_0^\infty\int_0^\infty x^z y^{1-z}e^{-x}e^{-y}d\mu(y)d\mu(x)=\int_0^\infty\int_0^\infty xy^{1-z}e^{-x(1+y)}d\mu(y)d\mu(x) \ .$$

Exchanging the order of integration and replacing $x\to\frac{x}{1+y}$ with $d\mu(\frac{x}{1+y})=d\mu(x)$

$$\Gamma(z)\Gamma(1-z)=\int_0^\infty\int_0^\infty xy^{1-z}e^{-x(1+y)}d\mu(x)d\mu(y)=\int_0^\infty\int_0^\infty\frac{x}{1+y}y^{1-z}e^{-x}d\mu(x)d\mu(y)$$

$$\Gamma(z)\Gamma(1-z)=\int_0^\infty\frac{y^{1-z}}{1+y}d\mu(y)\int_0^\infty xe^{-x}d\mu(x)=\int_0^\infty\frac{y^{1-z}}{1+y}d\mu(y)\cdot$$

Option 2: G. Polyá (20]) observed that

$$H(x) := 2\pi\sum_{1}^{\infty}(2\pi n^4 e^{-9x/2} - 3n^2 e^{-5x/2})e^{-n^2\pi e^{-2x}}$$

is asymptotically similar to

$$H(x) \approx H^*(x) := 8\pi^2 \cosh(\frac{9}{2}x)e^{-2\pi\cosh(2x)} \quad \text{as} \quad x \to \pm\infty$$

and proved that

$$\Xi^*(z) = \int_{-\infty}^{\infty} H^*(x)e^{izx}dx = 2\int_{0}^{\infty} H^*(x)\cos(zx)dx = 4\pi^2\left[K_{\frac{iz}{2}-\frac{9}{4}}(2\pi) + K_{\frac{iz}{2}+\frac{9}{4}}(2\pi)\right]$$

has only real zeros.

Polya's underlying argument (using the infinite (Weierstrass) product representation), proved by using a difference equation in z for the modified Bessel function of the third kind, is

Lemma O2: If $-\infty < c < \infty$ and $G(z)$ is an entire function of genus 0 or 1 that assumes real values for real z, has only real zeros and has at least one real zero, then the function

$$G(z - ic) + G(z + ic)$$

also has only real zeros. Especially it holds that the function

$$F_{a,c}(z) := K_{i(z-ic)}(a) + K_{i(z+ic)}(a)$$

has only real zeros for $-\infty < c < \infty$ and $a > 0$.

Unfortunately $\Xi^*(z)$ is no approximation to $\Xi(z)$ ([26] E. C. Titchmarsh,10.1). This disadvantage might be overcome by an alternative function $\Xi^{**}(z)$ properly defined, using

$$\varphi_{2\nu}(x) := \frac{\pi}{2}\left[J_\nu^2(x) + Y_\nu^2(x)\right] = \frac{\pi}{2}\left|H_\nu^{(1)}(x)\right|^2 = \frac{2}{\pi}\left|K_\nu(xe^{\pm\frac{\pi i}{2}})\right|^2$$

as generating function. In [10] G. Gasper the reality of the zeros of $\Xi^*(z)$ is proven, analyzing integrals of squares of certain real-valued special functions.

Option 3: [20] G. Polyá obtained the following general theorem about zeros of the Fourier transform of a real function:

Lemma O3: Let $0 \le a < b \le \infty$ and let $g(x)$ be a strictly positive continuous function on (a,b) and differentiable there except possible at finitely many points. Suppose that

$$\alpha \le -x\frac{g'(x)}{g(x)} \le \beta$$

at every point of (a,b) where $g(x)$ is differentiable. Suppose further that the integral

$$G(s) := \int_0^\infty x^s g(x)\frac{dx}{x}$$

Is convergent for $a^* < \mathrm{Re}(s) < \beta^*$. Then all zeros ρ of $G(s)$ in this strip satisfy $\alpha \le \mathrm{Re}(\rho) \le \beta$.

The parameter 2ν might be chosen appropriately to achieve

$$-x\frac{g'(x)}{g(x)} = \frac{1}{2}$$

for e.g. (1.7), i.e. $g(x) := \varphi_{2\nu}^*(x)$.

We note (see appendix), that applying Polya's lemma to Müntz’s formula (1.8) yields no information about the location of non-trivial zeros of the Zeta function.

Option 4: Lemma O2 is re-formulated in [6] D. Bump et.al., that an operator, which takes an even function $q(\nu)$ and replaces it by $\frac{q(\nu+1)-q(\nu-1)}{\nu}$ has the property of moving the zeros of a function closer on the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis:

If one restricts to $1/2 < \mathrm{Re}(s) =: \sigma < 1$ and put

$$2\nu := \varphi(s) := \mathrm{Re}(s-\sigma) \quad \text{it holds} \quad 0 < 2\nu < 1 \ .$$

Assuming that

$$\xi^*(s) := \zeta(s)\int_0^\infty x^s \varphi_{2\nu}(x)\frac{dx}{x} = \int_0^\infty x^s\left[\varphi_{2\nu\sigma}^*(x) - \frac{1}{x}\int_0^\infty \varphi_{2\nu}(t)dt\right]\frac{dx}{x}$$

has at least one zero might lead to a contradiction, taking into account that it holds

$$\varphi_{2\nu}(x) > 0 \quad , \quad \frac{d}{dx}\left[x\varphi_{2\nu}(x)\right] > 0 \quad , \quad \int_0^\infty \varphi_{2\nu}(t)dt = \frac{1}{2\pi}\cos(\frac{\pi}{2}2\nu)\int_0^\infty\int_0^\infty\int_0^\infty e^{-2t\sinh\rho\cosh\tau}\cosh[(2\nu)\tau]d\tau d\rho dt \ .$$

Option 5: Link the Bessel polynomials appropriately with **Ramanujan's Master Theorem** ([1] B. C. Berndt, The first quarterly report, 1.2 Theorem I), i.e.

$$\int_0^\infty F(x)x^{s-1}dx = \Gamma(s)\varphi(-s) \quad \text{for } F(x) = \sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k \text{ in the neighborhood of } x = 0$$

taking advantage of its properties given in lemma 0.5.

Motivation With $\varphi(k) := \frac{1}{\zeta(2k+1)}$ the Hardy/Littlewood resp. the Riesz equivalence criteria of the Riemann Hypothesis are

(5.1) RH holds if and only if $F(x) = \sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k = O(x^{-1/4})$.

(5.2) RH holds if and only if $\sum_1^\infty \frac{(-1)^{k+1}}{(k-1)!\zeta(2k)}x^k = O(x^{1/4+\varepsilon})$.

Ramanujan motivated his formula with the following wordings ([1] B. C. Berndt, chapter 4, Entry 8):

"<u>Statement</u>: If two functions of x be equal, then a general theorem can be formed by simply writing $\varphi(n)$ instead of x^n in the original theorem

<u>Solution</u>: "Put $x = 1$ and multiply it by $f(0)$ then change x to x, x^2, x^3, x^4.... and multiply $\frac{f'(0)}{1!}, \frac{f''(0)}{2!}, \frac{f'''(0)}{3!}$,... respectively and add up all the results. Then instead of x^n we have $f(x^n)$ for positive as well as for negative values of n . Changing $f(x^n)$ to $\varphi(n)$ we can get the result."

Example: $$\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$$

Ramanujan's building process:

$$f(0)[\arctan 1 + \arctan 1] = \frac{\pi}{2} f(0) , \quad \frac{f'(0)}{1!}\left[\arctan x + \arctan\frac{1}{x}\right] = \frac{f'(0)}{1!}\frac{\pi}{2} , \quad \frac{f''(0)}{2!}\left[\arctan x + \arctan\frac{1}{x}\right] = \frac{f''(0)}{2!}\frac{\pi}{2} \cdots$$

Replace $\arctan z$ by its Maclaurin series in z , where z is any integral power of x . Now add all the equalities above. On the left side one obtains two double series. Invert the order of summation in each double series to find that

$$\sum_0^\infty (-1)^n \frac{f(x^{2n+1}) + f(x^{-2n-1})}{2n+1} = \frac{\pi}{2} f(1) .$$

Replace $f(x^n)$ by $\varphi(n)$ to conclude that

$$\sum_0^\infty (-1)^n \frac{\varphi(2n+1) + \varphi(-2n-1)}{2n+1} = \frac{\pi}{2}\varphi(0) .$$

Of course, this formal procedure is fraught with numerous difficulties, but the theorem was finally correctly proved by G.H. Hardy.

Option 6 (just recalls known criteria)

There is a “modified” Zeta function representation $\xi^*(s) := \zeta(s)\tilde{\Pi}^*(s)$, which can be realized

either i) as a “convolution” $\xi^*(1/2+it) = (G*dF)(t) = \int_{-\infty}^{\infty} G(z-iu)dF(u)$

or ii) as “Fourier integral” $\xi^*(z) = \int_0^{\infty} u^{1-z}F(u)\varphi(\log u)\frac{du}{u}$

where $g(z) := \int_0^{\infty} u^{1-z}F(u)\frac{du}{u}$ has all its zeros on the critical line

Appendix

Lemma 0.2: For $\varphi_{2\nu}(x)$ and $\operatorname{Re}(x)>0$ the following asymptotic expansion holds true

$$\frac{1}{x}\varphi_{2\nu}(\frac{1}{x})=\sum_0^\infty P_{n.2\nu}x^{2n} \ .$$

Proof of Lemma 0.2: From the expansion

(A.1) $$\cosh 2\nu t=\cosh t\sum_0^\infty \frac{n!(\nu,n)}{(2n)!}2^{2n}\sinh^{2m} t=\cosh t\sum_0^\infty a_{\nu,m}\sinh^{2m} t$$

given in [26] G.N. Watson 7-4, 13-75, we get by substituting the variables first by $u=\sinh t$ and then by $y=2xu$ the identities

$$\varphi_{2\nu}(x)=\frac{4}{\pi}\int_0^\infty K_0(2x\sinh t)\cosh(2\nu t)dt=\frac{4}{\pi}\int_0^\infty K_0(2x\sinh t)\cosh t(\sum_0^\infty a_{\nu,m}\sinh^{2m} t)dt$$

$$\varphi_{2\nu}(x)=\frac{4}{\pi}\int_0^\infty uK_0(2xu)(\sum_0^\infty a_{\nu,n}u^{2n})\frac{du}{u}=\frac{4}{\pi}\sum_0^\infty a_{\nu,n}\int_0^\infty u^{2n+1}K_0(2xu)\frac{du}{u}$$

$$\varphi_{2\nu}(x)=\frac{4}{\pi}\sum_0^\infty a_{\nu,n}\int_0^\infty\left[\frac{y}{2x}\right]^{2n+1}K_0(y)\frac{dy}{y}=\frac{2}{\pi}\frac{1}{x}\sum_0^\infty 2^{-2n}a_{\nu,m}x^{-2n}\int_0^\infty y^{2n+1}K_0(y)\frac{dy}{y}$$

From lemma 0.1 we get

$$2^{2n-1}\Gamma^2(n+\frac{1}{2})=\int_0^\infty y^{2n+1}K_0(y)\frac{dy}{y}$$

resulting into

$$\varphi_{2\nu}(x)=\frac{1}{\pi x}\sum_0^\infty a_{\nu,n}x^{-2n}\Gamma^2(n+\frac{1}{2}) \ .$$

Using the formulas

(A.2) $$\frac{(2n)!}{2^{2n}n!}=\frac{\Gamma(1/2+n)}{\sqrt{\pi}}=\frac{(-1)^n}{\sqrt{\pi}\Gamma(1/2-n)} \quad \text{i.e.} \quad \Gamma^2(\frac{1}{2}+n)\Gamma^2(\frac{1}{2}-n)=\pi^2 \ .$$

it follows

$$\left[\frac{(2n)!}{2^{2n}n!}\right]^2\pi=\frac{\Gamma(1/2+n)}{\sqrt{\pi}} \ ,$$

i.e. $$\varphi_{2\nu}(x)=\frac{1}{x}\sum_0^\infty a_{\nu,n}x^{-2n}\left[\frac{(2n)!}{2^{2n}n!}\right]^2=\frac{1}{x}\sum_0^\infty\frac{(2n)!(\nu,n)}{2^{2n}n!}x^{-2n} \ . \bullet$$

We note the special cases

a) $P_{n.0} := (-1)^n \frac{[1*3*...(2n-1)]^4}{2^{2n}}$ for $2\nu := 0$

b) $P_{n.1/2} := (-1)^n \frac{(2n)!}{2^{3n} n!} *(1^2-(\frac{1}{2})^2)(3^3-(\frac{\sqrt{3}}{2})^2)...((2n-1)^3-(\frac{\sqrt{2n-1}}{2})^2)$ for $2\nu := 1/2$.

Proof of lemma 0.3 For $n \in N$ it holds Stirling formula $n! = n^{\frac{1}{2}+n} e^{-n} c_n$

whereby $\gamma_n := 1 - \frac{1}{12}\sum_1^{n-1}\frac{1}{\omega_k^2}$, $k \le \omega_k \le k+1$, $c_n = e^{\gamma_n}$

$$\lim \gamma_n = \gamma \quad , \quad \lim \frac{c_n^2}{c_{2n}} = \sqrt{2\pi} = e^{\gamma} \quad , \quad \lim \frac{\sqrt{n}}{n!}\frac{(2n)!}{2^{2n} n!} = \frac{1}{\sqrt{\pi}} \quad , \quad n \to \infty .$$

Putting $z := -\frac{1}{2}$ resp. $1 - z = \frac{1}{2} - n$ with $\sin \pi(\frac{1}{2}+n) = (-1)^n$ into the two formulas

$$\Gamma(z+n+1) = (s+n)(s+n-1).....(s+1)\Gamma(s+1) \quad \text{and} \quad \Gamma(z)\Gamma(1-z)\sin \pi z = \pi$$

leads to

$$\Gamma(\frac{1}{2}+n) = (n-\frac{1}{2})(n-\frac{3}{2}).....\frac{1}{2}\Gamma(\frac{1}{2}) \quad \text{and} \quad \Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n)\sin \pi(\frac{1}{2}+n) = \pi ,$$

which completes the proof of lemma 0.3. $\bullet$

Proof of lemma 0.5:

i) $P_{n.2\nu} := \frac{1*3*...(2n-1)}{2^n}(\nu,n) \propto \frac{p_{n,2\nu}}{2^{2n}\sqrt{\pi n}}$ follows from Stirling's formula

ii) $p_{n,2\nu} = (-1)^n [1*3*...(2n-1)]^2$, $P_{n.0} := (-1)^n \frac{[1*3*...(2n-1)]^4}{(2n)! 2^{2n}}$

ii) $P_{n.2\nu} = \frac{1}{\sqrt{\pi}}\Gamma(\frac{1}{2}+n)(\nu,n) = \frac{(-1)^n}{\pi}\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2}-n)}\frac{p_{n,2\nu}}{(2n)!}$ follows from lemma 0.3

iv) $p_{n,2\nu} = ((2\nu)-1)((2\nu)-3)...((2\nu)-(2n-1))((2\nu)+1)((2\nu)+3)...((2\nu)+(2n-1))$

$$p_{-n,2\nu} = ((2\nu)+1)((2\nu)+3)...((2\nu)+(2n-1))((2\nu)+(2n+1))((2\nu)-1)((2\nu)-3)...((2\nu)-(2n-1))((2\nu)-(2n+1))$$

and therefore $p_{-n,2\nu} = ((2\nu)^2 - (2n+1)^2) p_{n,2\nu}$

v) evident.

Proof of Lemma 1.1

i) from [11] I.S. Gradshteyn, I.M. Ryzhik (6.518) we recall

$$\frac{\pi}{2}\left[J_\nu^2(x)+Y_\nu^2(x)\right]=\frac{4\cos\nu\pi}{\pi}\int_0^\infty K_{2\nu}(2x\sinh t)dt \qquad \text{for} \quad -1<\mathrm{Re}(2\nu)<1\ ,\ \mathrm{Re}(x)>0$$

and from [27] G. N. Watson, 6-3 the formula

$$K_{2\nu}(2x\sinh t)=\int_0^\infty e^{-2x\sinh t\cosh\tau}\cosh(2\nu\tau)d\tau\,,$$

which gives $\varphi_{2\nu}(x)$ as Neumann-Nicholson integrals in the form

$$\varphi_{2\nu}(x)=\frac{4\cos\pi\nu}{\pi}\int_0^\infty\int_0^\infty e^{-2x\sinh t\cosh\tau}\cosh(2\nu\tau)d\tau dt\ .$$

With $\quad K_0(\xi)=\int_0^\infty e^{-\xi\cosh t}dt \quad , \quad K_{1/2}(x)=\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}$

it especially holds

$$\varphi_0(x)=\frac{1}{2\pi}\int_0^\infty K_0(2x\sinh t)dt=\frac{1}{2\pi}\int_0^\infty\int_0^\infty e^{-2x\sinh t\cosh\tau}d\tau dt$$

and

$$\varphi_{1/2}(x)=\frac{1}{2\pi\sqrt{2}}\int_0^\infty K_{1/2}(2x\sinh t)dt=\frac{1}{2\pi\sqrt{2}}\sqrt{\frac{\pi}{2}}\int_0^\infty\frac{e^{-2x\sinh t}}{\sqrt{2x\sinh t}}dt=\frac{1}{4\sqrt{2\pi}}\int_0^\infty\frac{e^{-2x\sinh t}}{\sqrt{x\sinh t}}dt\ .$$

i.e. $\quad \sqrt{\pi}\varphi_{\pm 1/2}(x)=\frac{1}{4}\int_0^\infty\frac{e^{-2x\sinh t}}{\sqrt{2x\sinh t}}dt \quad ,$

ii) $\quad \varphi_{2\nu}(x)d\Psi_\nu=\frac{\pi}{2}\left[J_\nu^2(x)+Y_\nu^2(x)\right]d\Psi=\frac{dx}{x} \quad$ with $\quad \Psi_\nu(x):=\arctan\left[\frac{Y_\nu(x)}{J_\nu(x)}\right],$

This relation to Hankel functions

$$H_\nu^{(1)}(x):=J(x)+iY_\nu(x)=R_\nu(x)e^{i\Psi_\nu}$$

with

$$R_\nu^2(x):=\left[J_\nu^2(x)+Y_\nu^2(x)\right]=\left|H_\nu^{(1)}(x)\right|^2\ ,\quad \Psi_\nu(x):=\arctan\left[\frac{Y_\nu(x)}{J_\nu(x)}\right]$$

and

$$\Psi_\nu'(x)=\frac{Y_\nu'(x)J_\nu(x)-Y_\nu(x)J_\nu'(x)}{J_\nu^2(x)+Y_\nu^2(x)}=\frac{(2/\pi x)}{J_\nu^2(x)+Y_\nu^2(x)}$$

is given by (see [26] G.N. Watson 13-73, 13-74, 15-8)

$$\varphi_{2\nu}(x)d\Psi_\nu = \frac{\pi}{2}\left[J_\nu^2(x) + Y_\nu^2(x)\right]d\Psi_\nu = \frac{dx}{x} \ .$$

iii) iv) are given [27] G.N. Watson 13-74 •

To prove proposition 1.3 we will use

Lemma A.2 It holds

i) $2\sqrt{\pi}\int_0^\infty \sinh^{-s} t dt = 2\sqrt{\pi}\int_0^\infty \sinh^{s-1} t dt = \Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2}) = \sqrt{\pi}B(\frac{1-s}{2},\frac{s}{2})$ for $0 < \mathrm{Re}(s) < 1$

ii) $\int_0^\infty \cosh^{-s} x dx = \frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{1+s}{2})} = \frac{1}{2}B(\frac{1}{2},\frac{s}{2})$ for $\mathrm{Re}(s) > 0$

iii) $\int_0^\infty \frac{\cosh \rho x}{\cosh^s x} dx = 2^{-s}B(\frac{s+\rho}{2},\frac{s-\rho}{2})$ for $\mathrm{Re}(s \pm \rho) > 0$

iv) $\int_0^\infty \frac{\sinh^\mu x}{\cosh^\nu x} dx = \frac{1}{2}B(\frac{1+\mu}{2},\frac{\nu-\mu}{2})$ for $-1 < \mathrm{Re}(\mu) < \mathrm{Re}(\nu)$

v) $\int_0^\infty \frac{\cos 2\xi t}{\cosh^{2\nu} t} dt = \frac{\sqrt{\pi}}{2}\frac{\left|\Gamma(\nu + i\xi)\right|^2}{\Gamma(\nu)\Gamma(\nu + 1/2)}$ for $\nu > 0$

vi) $\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} dx = \frac{1}{2}B(\frac{\mu}{2},\frac{\nu}{2})$ for $\mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$.

Proof of lemma A.2 From [11] I.S. Gradshteyn, I.M. Ryzhik (3.512) 1.& 2. we get

i) *For* $\mathrm{Re}(\mu) > -1$ *and* $\mathrm{Re}(\mu - \rho) < 0$ *it holds*

$$\Gamma(\frac{\rho+1}{2})\int_0^\infty \frac{\sinh^\mu x}{\cosh^\rho x} dx = \frac{1}{2}\Gamma(\frac{\mu+1}{2})\Gamma(\frac{\rho-\mu}{2}) = \frac{1}{2}B(\frac{\mu+1}{2},\frac{\rho-\mu}{2})\Gamma(\frac{\rho+1}{2})$$

ii) *For* $\mathrm{Re}(\mu \pm \beta) > 0$ *and* $a, b > 0$ *it holds* $\int_0^\infty \frac{\cosh 2bx}{\cosh^{2\mu} ax} dx = \frac{4^{\nu-1}}{a}B(\mu + \frac{b}{a}, \mu - \frac{b}{a})$.

Putting $\mu = -s$ and $\nu = 0$ results into $0 < \mathrm{Re}(s) < 1$ and

$$\int_0^\infty \sinh^{-s} x dx = \frac{1}{2}\frac{1}{\sqrt{\pi}}\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2}) = \frac{\cos\frac{\pi}{4}}{\sqrt{2\pi}}\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2}) = B(\frac{1-s}{2},\frac{s}{2}) \quad ,$$

which gives Lemma A.2 i)

Putting $\mu = 0$ and $\nu = -s$ gives Lemma A.2 ii)

Putting $2b := \rho$ $2\mu := s$, $a := 1$ gives $\mathrm{Re}(\mu \pm \beta) = \mathrm{Re}(\frac{s \pm \rho}{2}) > 0$ and lemma 1.2 iii)

iv) is given in [1] B. C. Berndt, quarterly reports 3.13

v) From [11] I.S. Gradshteyn, I.M. Ryzhik (3.621) we get

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} dx = \frac{1}{2} B(\frac{\mu}{2}, \frac{\nu}{2}) \quad , \quad \mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0 \qquad \bullet$$

Proof of Proposition 1.4 i) is given in [27] G. N. Watson, 13-74

ii) Applying the variable substitution $y = x \sinh t \cosh \tau$ i.e. $x^s = y^s [\sinh t \cosh \tau]^{-s}$ it follows

$$\varphi_{2\nu}(x) = \frac{\cos \pi\nu}{2\pi} \int_0^\infty \int_0^\infty e^{-2x \sinh t \cosh \tau} \cosh(2\nu\tau) d\tau dt$$

$$\int_0^\infty x^s \varphi_{2\nu}(x) \frac{dx}{x} = \frac{\cos \pi\nu}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty x^s e^{-x \sinh t \cosh \tau} \cosh(2\nu\tau) d\tau dt \frac{dx}{x} , \qquad \frac{dx}{x} = \frac{dy}{y}$$

$$= \frac{\cos \pi\nu}{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty y^s e^{-y} [\sinh t \cosh \tau]^{-s} \cosh(2\nu\tau) d\tau dt \frac{dy}{y}$$

$$= \Gamma(s) \frac{\cos \pi\nu}{2\pi} \int_0^\infty [\sinh t]^{-s} dt \int_0^\infty [\cosh \tau]^{-s} \cosh(2\nu\tau) d\tau$$

$$= \Gamma(s) \frac{\cos \pi\nu}{2\pi} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \Gamma(\frac{1-s}{2}) \Gamma(\frac{s}{2}) \int_0^\infty [\cosh \tau]^{-s} \cosh(2\nu\tau) d\tau$$

$$= \Gamma(s) \frac{\cos \pi\nu}{4\pi\sqrt{\pi}} \Gamma(\frac{1-s}{2}) \Gamma(\frac{s}{2}) \int_0^\infty \frac{\cosh(2\nu\tau)}{[\cosh \tau]^s} d\tau = \frac{\cos \pi\nu}{4\pi} \Gamma(s) B(\frac{1-s}{2}, \frac{s}{2}) \int_0^\infty \frac{\cosh(2\nu\tau)}{[\cosh \tau]^s} d\tau \quad .$$

Wit $\beta := \nu$, $2\mu := s$, $a := 1$ *and* $\mathrm{Re}(\frac{s}{2} \pm \nu) > 0$, lemma 1.2 iii) it therefore holds

$$= \frac{\cos \pi\nu}{4\pi} \Gamma(s) B(\frac{1-s}{2}, \frac{s}{2}) 4^{-\frac{s}{2}} B(\frac{s}{2} + \nu, \frac{s}{2} - \nu) \quad .$$

ii) with $\varphi_{1/2}(x) = \frac{1}{4\sqrt{2\pi}} \int_0^\infty \frac{e^{-2x \sinh t}}{\sqrt{x \sinh t}} dt$ it follows

$$\int_0^\infty x^s \varphi_{1/2} \frac{dx}{x} = \int_0^\infty \int_0^\infty x^{s-1/2} \frac{e^{-x \sinh t}}{\sqrt{\sinh t}} dt \frac{dx}{x} = \int_0^\infty \int_0^\infty y^{s-1/2} [\sinh t]^{1/2-s} \frac{e^{-y}}{\sqrt{\sinh t}} dt \frac{dy}{y} = \int_0^\infty \int_0^\infty [\sinh t]^{-s} dt y^{s-1/2} e^{-y} \frac{dy}{y}$$

$$= \frac{1}{4\sqrt{\pi}} \int_0^\infty \int_0^\infty x^{s-1/2} \frac{e^{-2x \sinh t}}{\sqrt{2x \sinh t}} dt \frac{dx}{x} = \frac{1}{4\sqrt{\pi}} \int_0^\infty \int_0^\infty y^{s-1/2} [2 \sinh t]^{1/2-s} \frac{e^{-y}}{\sqrt{\sinh t}} dt \frac{dy}{y}$$

$$=\frac{1}{4\sqrt{\pi}}2^{1/2-s}\int_0^\infty\int_0^\infty[\sinh t]^{-s}\,dt y^{s-1/2}e^{-y}\frac{dy}{y}=\sqrt{\frac{2}{\pi}}2^{-2-s}\int_0^\infty\int_0^\infty[\sinh t]^{-s}\,dt y^{s-1/2}e^{-y}\frac{dy}{y}\cdot$$

With lemma A.2 it follows for $0<\mathrm{Re}(s)<1$

$$\int_0^\infty x^s\varphi_{1/2}(x)\frac{dx}{x}=\sqrt{\frac{2}{\pi}}2^{-3-s}B(\frac{1-s}{2},\frac{s}{2})\int_0^\infty y^{s-1/2}e^{-y}\frac{dy}{y}=\sqrt{\frac{2}{\pi}}2^{-3-s}B(\frac{1-s}{2},\frac{s}{2})\Gamma(s-\frac{1}{2})$$

Applying the formula

$$\frac{2^{s-1}}{\sqrt{2\pi}}\Gamma(\frac{s+1/2}{2})\Gamma(\frac{s-1/2}{2})=\Gamma(s-\frac{1}{2})$$

then gives iii) •

Proof of Proposition 1.5

$$\int_0^\infty x^s\frac{d}{dx}[x\varphi_{2\nu}(x)]\frac{dx}{x}=\int_0^\infty x^s\frac{4}{\pi}\int_0^\infty K_0(2x\sinh t)\tanh t\cosh(2\nu t)\Theta_{2\nu}(t)dt\frac{dx}{x}$$

$$=\frac{4}{\pi}2^{-s}\int_0^\infty y^s\int_0^\infty K_0(y)\sinh^{-s}t\tanh t\cosh(2\nu t)\Theta_{2\nu}(t)dt\frac{dy}{y}$$

$$\int_0^\infty y^sK_0(y)\frac{dy}{y}=2^{s+2}\Gamma^2(s/2)$$

$$=\frac{16}{\pi}\Gamma^2(s/2)\int_0^\infty\sinh^{-s}t\tanh t\cosh(2\nu t)[\tanh t-2\nu\tanh 2\nu t]dt$$

$$=\frac{16}{\pi}\Gamma^2(s/2)\int_0^\infty\left[\frac{\cosh(2\nu t)}{\cosh^2t\sinh^{s-2}t}-\frac{2\nu\sinh(2\nu t)}{\cosh t\sinh^{s-1}t}\right]dt$$

$$=\frac{16}{\pi}\Gamma^2(s/2)\int_0^\infty\frac{1}{\cosh^2t\sinh^{s-2}t}[\cosh(2\nu t)-2\nu\sinh t\cosh t\sinh(2\nu t)]dt$$

$$=\frac{16}{\pi}\Gamma^2(s/2)\int_0^\infty\frac{1}{\cosh^2t\sinh^{s-2}t}\left[\cosh t\sum_0^\infty a_{\nu,m}\sinh^{2m}t-2\nu\sinh t\cosh t\sinh(2\nu t)\right]dt$$

$$=\frac{16}{\pi}\Gamma^2(s/2)\int_0^\infty\frac{1}{\cosh t\sinh^{s-2}t}\left[\sum_0^\infty a_{\nu,m}\sinh^{2m}t-2\nu\sinh t\sinh(2\nu t)\right]dt$$

$$=\frac{16}{\pi}\Gamma^2(s/2)\left[\int_0^\infty\sum_0^\infty a_{\nu,m}\frac{\sinh^{2m-s+2}t}{\cosh t}dt-2\nu\int_0^\infty\frac{\sinh^{3-s}t\sinh(2\nu t)}{\cosh t}dt\right]$$

$$\int_0^\infty\frac{\sinh^{2m-s+2}t}{\cosh t}dt=\frac{1}{2}\Gamma(m-\frac{s-3}{2})\Gamma((1-m)+\frac{s-3}{2})$$

$$\frac{\sinh^{3-s}t\sinh(2\nu t)}{\cosh t}=\frac{\sinh^{3-s}t\sinh t}{\cosh t}=\frac{\sinh^{4-s}t}{\cosh t}$$

$$2\nu\int_0^\infty \frac{\sinh^{3-s} t \sinh(2\nu t)}{\cosh t}dt = \int_0^\infty \frac{\sinh^{4-s} t}{\cosh t}dt = \frac{1}{2}\Gamma(1-\frac{s-3}{2})\Gamma(\frac{s-3}{2})$$

$$= \frac{8}{\pi}\Gamma^2(\frac{s}{2})\left[\sum_0^\infty a_{\frac{s}{2},m}\Gamma(m-\frac{s-3}{2})\Gamma((1-m)+\frac{s-3}{2}) - \Gamma(1-\frac{s-3}{2})\Gamma(\frac{s-3}{2})\right] \qquad \bullet$$

Lemma (**Müntz' formula**) For $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$ it holds

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1.$$

Proof: i) because $\omega(x)$ is continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ it holds

$$\sum_1^\infty \frac{1}{n^s}\left|\int_0^\infty x^{s-1}\omega(x)\right|dx \qquad \text{exists for } 1 < \sigma < \alpha \text{ ,}$$

i.e. the inversion leading to the left hand side of (4.3) is justified.

ii) $$\sum_1^\infty \omega(nx) - \int_0^\infty \omega(xt)dt = x\int_0^\infty \omega'(t)(t-[t])dt = x\int_0^{1/x} O(1)dt + x\int_{1/x}^\infty O((xt)^{-\beta})dt = O(1)$$

The first summand is justified, because $\omega(x)$ is continuous and bounded in any finite interval the second summand is justified, because $\omega(x) = o(x^{-\alpha})$, i.e. it holds

$$\sum_1^\infty \omega(nx) = O(1) + \frac{c}{x} \quad \text{with} \quad c = \int_0^\infty \omega(t)dt \,.$$

Hence $$\int_0^\infty x^s \sum_1^\infty \omega(nx) + \frac{dx}{x} = \int_0^1 x^s \left[\sum_1^\infty \omega(nx) - \frac{c}{x}\right]\frac{dx}{x} + \int_1^\infty x^s \sum_1^\infty \omega(nx)\frac{dx}{x} + \frac{c}{s-1}$$

for $\sigma > 0$ except $s = 1$. Also

$$-c\int_1^\infty x^{s-2}dx = \frac{c}{s-1} \qquad \text{for } \sigma < 1$$

and therefore (4.3) for $0 < \sigma = \mathrm{Re}(s) < 1$ $\bullet$

Lemma (Polya's criterion) A real self-adjoint operator of the form

$$f(z) \to \int_a^{1/a} x^{-z} G(x) \frac{dx}{x} \qquad \text{(i.e. } G(x) \text{ real and } G(x) = G(\frac{1}{x})\text{)},$$

which has the property that $G(x)$ is non-decreasing on the line $[1, a]$ has the property that the zeros of its transform all lie on the line $\mathrm{Re}(z) = 0$.

Polya's criterion is given in [8] H.M. Edwards, 12.5.

Remark The lemma above is valid for an integral over a finite interval and to extend it to an infinite interval it needs certain conditions. In order to apply Polya's criterion directly to Müntz' formula

$$\zeta(s)\int_0^\infty x^s \frac{\omega(x)dx}{x} = \int_0^\infty x^s \left[\sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt\right]\frac{dx}{x} \qquad \text{for } 0 < \mathrm{Re}(s) < 1.$$

for $\omega(x), \omega'(x)$ continuous and bounded in any finite interval with $\omega(x) = o(x^{-\alpha})$ and $\omega(x) = o(x^{-\beta})$ for $x \to \infty$ and $\alpha, \beta > 1$, one needs to show that the function

$$\text{(*)} \qquad P(x) = \sum_1^\infty \omega(nx) - \frac{1}{x}\int_0^\infty \omega(t)dt$$

is positive and increasing for $x > 0$. However, $P(x)$ cannot be positive and increasing in the whole range for $x > 0$, because otherwise its value at infinity would be positive and not 0, as is the case, because Muentz's formula requires $\omega(x)$ to vanish at infinity to order $x^{-\alpha}$ with $\alpha > 1$. Hence the function $P(x)$ vanishes to at least order $\frac{1}{2}$ and in particular it has the value 0 at infinity. Therefore, the expression in the form (*) cannot be both positive and increasing near infinity, i.e. Polya's criterion in form of lemma 0.2 never applies to a formula of Muentz's type.

Lemma (***Theorem of Frullani***) Let $f(x)$ be a continuous, integrable function over any interval $0 \le A \le x \le B < \infty$. Then, for $0 < b < a$,

$$\int_0^\infty [f(ax) - f(bx)]\frac{dx}{x} = [f(\infty) - F(0)]\log\frac{a}{b}$$

where $f(0) = \lim f(x)$ for $x \to 0^+$ and $f(\infty) = \lim f(x)$ for $x \to \infty$.

We mention a generalization of this lemma, due to Hardy (Quart. J. Math. 33 (1902) p. 113-144) in the form

$$\int_0^\infty [\varphi(ax^m) - \psi(bx^n)](\log x)^p \frac{dx}{x}.$$

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