PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 2, October 1987

ZEROS OF DERIVATIVES OF ENTIRE FUNCTIONS

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ABSTRACT. It is shown that if a real entire function of genus one has only finitely many nonreal zeros, then, its derivatives, from a certain one onward, have only real zeros.

A real entire function $\psi(x)$ is said to be in the *Laguerre-Pólya class* if $\psi(x)$ can be expressed in the form

$$\psi(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} (1 + x/a_k)e^{-x/a_k},$$

where c, β, a_k are real, $\alpha \geq 0$, n is a nonnegative integer, and $\sum a_k^{-2} < \infty$ (see [**L**, **P1**]). If $\psi(x)$ is in the Laguerre-Pólya class, we will write $\psi \in \mathcal{L}\text{-}\mathcal{P}$. Of particular importance is the fact that such a function can be uniformly approximated on compact subsets of the complex plane by a sequence of polynomials with only real zeros. We shall use the notation $\mathcal{L}\text{-}\mathcal{P}^*$ to denote the set of all entire functions which arise as products of real polynomials and functions in $\mathcal{L}\text{-}\mathcal{P}$.

A fifty-five year old conjecture of Pólya [**P2**] and Wiman [**W1**] states that the derivatives $\varphi^{(n)}(x)$ for $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}^*$ will have only real zeros for all sufficiently large n. All work on this problem depends heavily on the order of $\varphi(x)$. (The corresponding statement fails for some functions of order 2 such as $\exp(x^2)$.) The first partial result was proved by Ålander in 1930 for functions of order less than $\frac{2}{3}$ [**A2**] and later extended by Wiman to functions of order at most 1 [**W2**]. In 1937, Pólya extended these results to functions of order less than $\frac{4}{3}$ [**P3**]. Recently, the present authors have proved the conjecture for functions of order less than 2 [**CCS**]. In this paper, we first obtain a refinement of an old theorem of Ålander [**A1**] which was needed in [**A2**] and [**CCS**]. We use this improved result to extend our proof of the Pólya-Wiman conjecture to functions of minimal type of order 2 (for definitions, see [**B**]).

THEOREM 1. *Suppose that $f(z) = \sum b_k z^k$ is a transcendental entire function satisfying*

$$M(r) = \max_{|z|=r} |f(z)| \leq e^{cr^d} \quad \textit{for all } r \geq r_0,$$

where c and d are positive constants. Let $\varepsilon > 0$. Then there are infinitely many positive integers n such that if $f^{(n)}(z_n) = 0$, then

$$(1) \qquad |z_n| > (\log 2)e^{-1}(c+\varepsilon)^{-1/d} n^{-1+1/d}.$$

Received by the editors June 30, 1986. Presented at the 832nd meeting of the AMS in Honolulu, Hawaii, March 26–28, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D15, 30D35.

Key words and phrases. Pólya-Wiman conjecture, Laguerre-Pólya class.

This theorem is an improvement of Ålander's theorem (see, for example, [**CCS**]). Ålander's theorem applied to $f(z)$ would yield, in place of (1), the inequality $|z_n| > (\log 2)n^{-1+1/\lambda}$ for any $\lambda > d$.

PROOF OF THEOREM 1. We have by Cauchy's inequality that

$$|b_n| \le \frac{e^{\log M(r)}}{r^n}, \qquad n \ge 0,\ r > 0. \tag{2}$$

Let $H(r)$ denote the inverse function of $\log M(r)$. It follows that for $r \ge cr_0^d = r_1$,

$$r^{1/d}c^{-1/d} \le H(r). \tag{3}$$

If $r = H(n)$, then by (2), $|b_n| \le (e/H(n))^n$. Thus, by (3), there is a positive integer n_1 such that $|b_n| \le (en^{-1/d})^n c^{n/d}$ for $n \ge n_1$. In particular, if $\delta > 0$, then $|b_n| n^{n/d} e^{-n}(c+\delta)^{-n/d} = o(1)$ as $n \to \infty$. Hence there are arbitrarily large n such that

$$|b_{k+n}/b_n| < e^k (c+\delta)^{k/d}(k+n)^{-k/d}, \qquad k \ge 1. \tag{4}$$

Then by (4), there are infinitely many n such that

$$\begin{aligned}\left|\frac{f^{(n)}(z)}{n!b_n}\right| &= \left|\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{b_{k+n}}{b_n}z^k\right| \\ &\ge 1 - \sum_{k=1}^{\infty}\binom{n+k}{k}\left|\frac{b_{k+n}}{b_n}\right||z^k| \\ &> 1 - \sum_{k=1}^{\infty}\binom{n+k}{k}\frac{e^k(c+\delta)^{k/d}}{(n+k)^{k/d}}|z^k| \\ &> 2 - \sum_{k=0}^{\infty}\binom{n+k}{k}\frac{e^k(c+\delta)^{k/d}}{n^{k/d}}|z^k|.\end{aligned}$$

If $|z|e(c+\delta)^{1/d}n^{-1/d} < 1$, then the expression above is equal to

$$2 - (1 - en^{-1/d}(c+\delta)^{1/d}|z|)^{-n-1}. \tag{5}$$

Also, for sufficiently large n,

$$(\log 2)e^{-1}(c+3\delta)^{-1/d}n^{-1+1/d} < (\log 2)e^{-1}(c+2\delta)^{-1/d}n^{1/d}(n+1)^{-1}.$$

We conclude that if $|z| \le (\log 2)e^{-1}(c+3\delta)^{-1/d}n^{-1+1/d}$ and n is large, then expression (5) is larger than

$$2 - \left(1 - \frac{\log 2}{n+1}\left(\frac{c+\delta}{c+2\delta}\right)^{1/d}\right)^{-n-1}. \tag{6}$$

Since (6) is positive for all sufficiently large n, the theorem follows by setting $\delta = \varepsilon/3$.

We now present an example showing that Theorem 1 is essentially sharp. Let $f(z) = e^{-cz^2}$, where $c > 0$. Then $f^{(n)}$ is an odd function if n is odd, so $f^{(n)}(0) = 0$. Also, if n is even there is a zero z_n of $f^{(n)}$ satisfying

$$c^{-1/2}n^{-1/2} \le z_n \le (3/2)^{1/2}c^{-1/2}n^{-1/2}.$$

These estimates are due to Wiman [**W1**]. From this it is clear that the only possible improvement in inequality (1) would be an improvement in the constant $(\log 2)/e$.

An immediate consequence of Theorem 1 (with $d = 2$, ε replaced by $\varepsilon/2$, and $c = \alpha + \varepsilon/2$) for functions in the Laguerre-Pólya class is the following corollary.

COROLLARY 1. *Suppose that*

$$f(z) = p(z)e^{-\alpha z^2+\beta z}\prod(1 + z/a_k)e^{-z/a_k}$$

is a transcendental function in $\mathcal{L}\text{-}\mathcal{P}^*$, *where* $p(z)$ *is a real polynomial, and that* $\varepsilon > 0$. *Then there are infinitely many positive integers* n *such that if* $f^{(n)}(z_n) = 0$, *then*

$$|z_n| > (\log 2)e^{-1}(\alpha + \varepsilon)^{-1/2}n^{-1/2}.$$

We now turn our attention to the Pólya-Winman conjecture. We shall need the following results, in which we write $D^n f$ for $f^{(n)}$.

LEMMA 1 [**CCS**, LEMMA 1]. *If* $\varphi \in \mathcal{L}\text{-}\mathcal{P}^*$ *and if* $D^m\varphi \in \mathcal{L}\text{-}\mathcal{P}$ *for some nonnegative integer* m, *then for any* $a \in \mathbf{R}$

$$D^{m+1}[(x+a)\varphi(x)] \in \mathcal{L}\text{-}\mathcal{P}.$$

LEMMA 2 [**CCS**, LEMMA 2]. *Let* $\varphi(x) = p(x)\psi(x) \in \mathcal{L}\text{-}\mathcal{P}^*$, *where* $p(x)$ *is a nonconstant polynomial with only nonreal zeros and where*

$$\psi(x) = cx^n e^{\beta x}\prod_{k=1}^{\infty}(1 + x/a_k)e^{-x/a_k}$$

is in $\mathcal{L}\text{-}\mathcal{P}$. *Then there is a positive integer* N *and an open nonempty interval* I *such that if* $\gamma \in I$, *then* $(D + \gamma)\varphi_N(x)$ *has fewer nonreal zeros than* $p(x)$, *where*

$$\varphi_N(x) = cp(x)\left(\exp\left\{\left(\beta - \sum_{k=1}^{N-1}\frac{1}{a_k}\right)x\right\}\right)\prod_{k=N}^{\infty}\left(1 + \frac{x}{a_k}\right)e^{-x/a_k}.$$

THEOREM 2. *Let* $\varphi \in \mathcal{L}\text{-}\mathcal{P}^*$. *If* $\gamma_1 < \gamma_2$ *and if* $(D + \gamma_j)\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, $j = 1, 2$, *then*

$$(D + \gamma)\varphi(x) \in \mathcal{L}\text{-}\mathcal{P} \quad \text{for all } \gamma \in [\gamma_1, \gamma_2].$$

Moreover, the real zeros of $D(\varphi'/\varphi)$ *are all simple.*

This theorem, but with a restriction on the order of φ, is [**CCS**, Corollary 1]. The theorem is proved by using the same counting argument as in the original proof, except that a refinement of [**CCS**, Lemma 3] is now required. The upper bound in that lemma must be recognized to be $2d + 1$ when φ has infinitely many zeros.

THEOREM 3. *Let* $\varphi(x) = p(x)e^{\beta x}\prod(1 + x/a_k)e^{-x/a_k} \in \mathcal{L}\text{-}\mathcal{P}^*$, *where* $p(x)$ *is a polynomial. If* $(D + \gamma)\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ *for all* γ *in an open nonempty interval* I, *then there is a positive integer* m *such that* $D^m\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$.

This theorem but with a restriction to order less than 2, was proved as Theorem 2 of [**CCS**]. It was proved by obtaining a contradiction involving Ålander's theorem. The same proof yields Theorem 3 when we use Corollary 1 (with $\alpha = 0$ and ε sufficiently small) in place of Ålander's theorem. It is important to notice that Theorem 1 is translation invariant.

We now give our application of Theorem 1 to the Pólya-Wiman conjecture.

THEOREM 4. *If $p(x)$ is a polynomial and $\varphi(x) = p(x)e^{\beta x}\prod(1+x/a_k)e^{-x/a_k} \in \mathcal{L}\text{-}\mathcal{P}^*$, then there is a positive integer M such that $D^M\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$.*

PROOF. We may assume that $p(x)$ has no real zeros and has degree $2d$. If $d = 1$, we obtain the conclusion by applying Lemma 2, Theorem 3, and Lemma 1 in that order. For $d > 1$, we proceed by induction. Apply Lemma 2, obtaining $\gamma_1 < \gamma_2$ such that $(D + \gamma_j)\varphi_N$ have less than $2d$ nonreal zeros. By the induction hypothesis, there exists a number r such that $D^r(D+\gamma_j)\varphi_N$ is in $\mathcal{L}\text{-}\mathcal{P}$. By Theorem 2, $(D + \gamma)(D^r\varphi_N)$ is in $\mathcal{L}\text{-}\mathcal{P}$ for all γ in (γ_1, γ_2). Therefore by Theorem 3, there exists an m such that $D^{m+r}\varphi_N$ is in $\mathcal{L}\text{-}\mathcal{P}$. An appeal to Lemma 1 completes the proof.

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