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LECTURES ON THE THEORY OF ENTIRE FUNCTIONS

by

S.A. Buterin, G. Freiling and V.A. Yurko

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The main purpose of these lecture notes is to give a concise introduction to the theory of entire functions that covers in particular those topics that are used frequently in our own field of research - in particular in spectral theory. We assume that the reader knows the basic facts on functions of one complex variable as they can be found, for example, in the textbooks [1]–[3]. For additional results on entire functions and also on meromorphic functions we refer the reader to [4]-[10].

1. Entire functions

A function $f(z)$ of a complex variable z is called an *entire function* if it is analytic in the whole complex plane. An entire function $f(z)$ can be represented by its power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

with infinite radius of convergence. By virtue of the Cauchy-Hadamard formula [1], [2],

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}=0. \tag{1.2}$$

Clearly, each polynomial is an entire function. In this case only a finite number of the coefficients a_n in (1.1) differs from zero. If an entire function $f(z)$ is not a polynomial, i.e. if in (1.1) infinitely many coefficients a_n differ from zero, then it is called *transcendental.* In other words, an entire function is transcendental if and only if ∞ is an essential singularity. For example, the functions

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}, \quad \sin z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \cos z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!},$$

$$\cos\sqrt{z}=\sum_{n=0}^{\infty}\frac{(-1)^n z^{n}}{(2n)!}, \quad \frac{\sin\sqrt{z}}{\sqrt{z}}=\sum_{n=0}^{\infty}\frac{(-1)^n z^{n}}{(2n+1)!},$$

are entire transcendental ones.

Thus, entire functions form a simplest class of analytic functions that includes all polynomials. While a polynomial has a finite number of zeros, a transcendental entire function may have both finite and infinite number of zeros, and even no zero at all. Moreover, according to the Picard theorem [2], for each transcendental entire function $f(z)$ the function $f(z)-A$ has infinitely many zeros for all $A\in\mathbb{C}$ with at most one exceptional value. For example, for e^z this exceptional value is $A=0$ and for $\sin z$ there is no such value. It is known that a polynomial can be expanded into a finite product of linear factors. One of the important questions of the theory of entire functions is the expansion of a transcendental entire function possessing infinitely many zeros into an infinite product.

The polynomials can be classified by their growth, which has only power range. According to the fundamental theorem of algebra, the power n of any polynomial is firmly connected with the number of its zeros m, namely $n = m$. The connection between the growth and the distribution of zeros of entire functions is the main question of the theory of entire functions. The general rule states that the "more zeros" a nonzero entire function has, the bigger its global growth is. The fundamental characteristic of the global growth of an entire function $f(z)$ is the maximum of the modulus:

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

It follows from the maximum modulus principle for analytic functions that $M_f(r)$ increases monotonically. Clearly, for a polynomial of degree N one has

$$M_f(r) = O(r^N), \quad r \to \infty, \tag{1.3}$$

where O is the Landau symbol. However, the next theorem shows that the power function is not applicable for the estimation of growth of transcendental entire functions.

Theorem 1.1. *For each transcendental entire function the following relation holds*

$$\lim_{r\to\infty} \frac{\ln M_f(r)}{\ln r} = \infty \tag{1.4}$$

Proof. Suppose that

$$\varliminf_{r\to\infty} \frac{\ln M_f(r)}{\ln r} = \alpha < \infty,$$

Then for each $\alpha' > \alpha$, there exists a sequence $\{r_m\}$, which tends to infinity, and

$$\ln M_f(r_m) < \alpha' \ln r_m$$

for all m. Hence, by virtue of the Cauchy inequality, we have

$$|a_n| \le \frac{M_f(r_m)}{r_m^n} < r_m^{\alpha'-n}. \tag{1.5}$$

Since $r_m \to \infty$, it follows from (1.5) that $a_n = 0$ for $n > \alpha'$. Thus, $f(z)$ is a polynomial of degree at most $[\alpha]$. □

Corollary 1.1. *If (1.3) is valid for an entire function $f(z)$, then $f(z)$ is a polynomial of degree not greater than N.*

In particular, from this we derive the well-known Liouville theorem.

Theorem 1.2 (Liouville). *If the entire function $f(z)$ is bounded (i.e. $M_f(r) = O(1)$, $r \to \infty$), then $f(z) \equiv const$.*

Another important question of the theory of entire functions is devoted to the connection between the growth of an entire function $f(x)$ in different directions and its global growth characterized by $M_f(r)$. Unlike polynomials, which increase equally in all directions, transcendental entire functions may grow differently in different directions and in some directions they may even decrease. For example, for each $\varepsilon > 0$ the function e^z increases in $\{z : \arg z \in [-\pi/2+\varepsilon, \pi/2-\varepsilon]\}$ and decreases in $\{z : \arg z \in (-\pi, -\pi/2-\varepsilon] \cup [\pi/2+\varepsilon, \pi]\}$. The general rule says that if an entire function has "small global growth", then it cannot decrease in rather different directions and must increase in a sufficiently large part of $\mathbb{C}$.

2. The order and the type of an entire function

2.1. The order of an entire function. An entire function $f(z)$ is of *finite order*, if there exists $\mu > 0$ such that for sufficiently large r $(r > R)$,

$$M_f(r) < \exp(r^\mu). \tag{2.1}$$

The greatest lower bound of those μ, for which (2.1) holds, is called the *order* of the entire function $f(z)$, and is denoted by ρ, i.e. $\rho = \inf \mu \geq 0$. If (2.1) is not fulfilled for any finite μ, we shall say that $f(z)$ has *infinite order* $(\rho = \infty)$.

Theorem 2.1. *The order of an entire function is calculated by the formula*

$$\rho = \overline{\lim_{r\to\infty}} \frac{\ln\ln M_f(r)}{\ln r}. \tag{2.2}$$

Proof. By definition of the order ρ, for each $\varepsilon > 0$ the inequality

$$M_f(r) < \exp(r^{\rho+\varepsilon}) \tag{2.3}$$

holds for $r > R(\varepsilon)$. Moreover, there exist $r_n \to \infty$ such that

$$M_f(r_n) > \exp(r_n^{\rho-\varepsilon}). \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\frac{\ln\ln M_f(r)}{\ln r} < \rho + \varepsilon, \quad r > R(\varepsilon), \qquad \rho - \varepsilon < \frac{\ln\ln M_f(r_n)}{\ln r_n}.$$

Hence we arrive at (2.2). □

Formula (2.2) can be used for the definition of the order of an entire function $f(z)$.

Remark 2.1. It follows from (2.3) and (2.4) that

$$|f(z)| < \exp(|z|^{\rho+\varepsilon}), \quad |z| > R_0(\varepsilon), \qquad |f(z_n)| > \exp(|z_n|^{\rho-\varepsilon}), \tag{2.5}$$

where $z_n \to \infty$ ($\{z_n\}$ depends on f and ε), and $\varepsilon > 0$ is arbitrary.

Example 2.1. Let $f(z) = e^z$. Clearly, $|e^z| \leq e^{|z|}$. Moreover, for $z = r$ we have the equality. Hence, $M_f(r) = e^r$. Using (2.2) we get $\rho = 1$. Analogously one can check that the function $f(z) = \exp(z^m)$ has the order $\rho = m$.

Example 2.2. Let $f(z) = e^{e^z}$. Then $M_f(r) = e^{e^r}$, and consequently, this entire function is of infinite order: $\rho = \infty$.

2.2. The type of an entire function. Let the entire function $f(z)$ have the order ρ $(0 < \rho < \infty)$. We shall say that $f(z)$ is of *finite type* (with respect to the order ρ) if there exists $a > 0$ such that

$$M_f(r) < \exp(ar^\rho), \quad r > R. \tag{2.6}$$

The greatest lower bound of those values of a, for which (2.6) is fulfilled, is called the *type* of the entire function $f(z)$, and is denoted by σ. If (2.6) is not fulfilled for any finite a, we shall say that $f(z)$ is of *infinite type* with respect to the order ρ and write $\sigma = \infty$.

Theorem 2.2. *The type of an entire function* $f(z)$ *of order* ρ *is calculated by the formula*

$$\sigma = \varlimsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}. \tag{2.7}$$

Proof. We have

$$M_f(r) < \exp((\sigma+\varepsilon)r^\rho), \ \ r > R(\varepsilon), \qquad M_f(r_n) > \exp((\sigma-\varepsilon)r_n^\rho), \ \ r_n \uparrow \infty. \tag{2.8}$$

From (2.8) we derive (2.7). □

Formula (2.7) can be used for the definition of the type of an entire function $f(z)$.

Remark 2.2. From the inequalities (2.8) for $\sigma < \infty$, it follows that

$$|f(z)| < \exp((\sigma+\varepsilon)|z|^\rho), \ \ |z| > R(\varepsilon), \qquad |f(z_n)| > \exp((\sigma-\varepsilon)|z_n|^\rho), \tag{2.9}$$

where $\{z_n\}$ is a sequence which depends on f and $\varepsilon > 0$.

Example 2.3. Let

$$f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Obviously, $|\sin z| \le \frac{1}{2}(e^{|z|} + e^{|z|}) = e^{|z|}$. For $z = -ir$, $r > 0$ we have

$$|\sin(-ir)| = \frac{1}{2}(e^r - e^{-r}) \ > \ \frac{1}{2}e^r \ > \ e^{(1-\varepsilon)r}, \ \ r > r_\varepsilon.$$

Hence, for the function $\sin z$, one gets $\rho = 1$ and $\sigma = 1$.

We shall say that the entire function $f(z)$ of order ρ $(0 < \rho < \infty)$ is of *minimal, normal or maximal type*, if $\sigma = 0$, $0 < \sigma < \infty$ or $\sigma = \infty$, respectively.

2.3. Determination of the order and the type from the coefficients of the power series expansion. The order and the type of an entire function can be determined from the coefficients of the series (1.1). More presicely, the following theorem is valid.

Theorem 2.3 (Lindelöf, Pringsheim). *The order* ρ *of the entire function* $f(z)$ *of the form (1.1) is determined by the formula*

$$\rho = \varlimsup_{n\to\infty} \frac{n \ln n}{\ln \left|\dfrac{1}{a_n}\right|}. \tag{2.10}$$

If the entire function $f(z)$ *has the order* ρ $(0 < \rho < \infty)$, *then its type* σ *is determined by the formula*

$$(\sigma e \rho)^{1/\rho} = \varlimsup_{n\to\infty} n^{1/\rho} \sqrt[n]{|a_n|}. \tag{2.11}$$

The proof is based on the following two lemmas.

Lemma 2.1. *If* $M_f(r) < \exp(\beta r^\alpha)$, $r > R$, *then* $|a_n| < \left(\dfrac{e\alpha\beta}{n}\right)^{n/\alpha}$, $n > N$.

Proof. By virtue of the Cauchy inequality we have

$$|a_n| \le \frac{M_f(r)}{r^n} < \exp(\beta r^\alpha - n \ln r) =: g_n(r), \quad r > R. \tag{2.12}$$

The function $g_n(r)$ has a minimum for $r_n = \Big(n/(\beta\alpha)\Big)^{1/\alpha}$. Substituting $r = r_n$ into the estimate (2.12) we arrive at the assertion of the lemma. $\square$

Lemma 2.2. *If* $|a_n| < \Big(\frac{e\alpha\beta}{n}\Big)^{n/\alpha}$, $n > N$, *then for each* $\varepsilon > 0$ *there exists* R_ε *such that* $M_f(r) < \exp((\beta+\varepsilon)r^\alpha)$, $r > R_\varepsilon$.

Proof. Without loss of generality we assume that $a_n = 0$ for $n \le N$. Hence

$$M_f(r) \le \sum_{n=N+1}^{\infty} \Big(\frac{e\alpha\beta r^\alpha}{n}\Big)^{n/\alpha}. \tag{2.13}$$

Put $m_n := [n/\alpha]$. Since one can also assume that $e\beta r^\alpha \ge 1$ and $N \ge \alpha$, we have

$$(e\beta r^\alpha)^{n/\alpha} \le (e\beta r^\alpha)^{m_n+1}, \qquad \Big(\frac{\alpha}{n}\Big)^{n/\alpha} \le \Big(\frac{1}{m_n}\Big)^{m_n}.$$

Substituting this into (2.13) we get

$$M_f(r) \le e\beta r^\alpha \sum_{n=N+1}^{\infty} \Big(\frac{e\beta r^\alpha}{m_n}\Big)^{m_n}.$$

Obviously, we have $m_{n+[\alpha]+1} \ge m_n + 1$ and hence we arrive at

$$M_f(r) \le ([\alpha]+1)e\beta r^\alpha \sum_{n=1}^{\infty} \Big(\frac{e\beta r^\alpha}{n}\Big)^n. \tag{2.14}$$

According to the Stirling formula $n! \sim (n/e)^n\sqrt{2\pi n}$ we have

$$\Big(\frac{e}{n}\Big)^n \le \frac{C_\varepsilon}{n!}\Big(\frac{\beta+\varepsilon/2}{\beta}\Big)^n.$$

Substituting this into (2.14) we obtain

$$M_f(r) \le C_\varepsilon([\alpha]+1)e\beta r^\alpha \sum_{n=0}^{\infty} \frac{((\beta+\varepsilon/2)r^\alpha)^n}{n!}$$

$$= C_\varepsilon([\alpha]+1)e\beta r^\alpha \exp((\beta+\varepsilon/2)r^\alpha) < \exp((\beta+\varepsilon)r^\alpha), \quad r > R_\varepsilon.$$

$\square$

Proof of Theorem 2.3. (i) According to the definition of the order and Lemma 2.1 we get $|a_n| < \Big(\frac{e(\rho+\varepsilon)}{n}\Big)^{n/(\rho+\varepsilon)}$, $n > N_\varepsilon$, whence we get

$$\ln\frac{1}{|a_n|} > \frac{n}{\rho+\varepsilon}(\ln n - \ln e(\rho+\varepsilon)) = \frac{n\ln n}{\rho+\varepsilon}(1+o(1)), \quad n \to \infty.$$

By virtue of the arbitrariness of $\varepsilon > 0$ we arrive at

$$A := \varlimsup_{n\to\infty} \frac{n\ln n}{\ln\Big|\frac{1}{a_n}\Big|} \le \rho$$

On the other hand, according to Lemma 2.2 for each $\varepsilon > 0$ there exists an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $\left(\dfrac{e(\rho-\varepsilon)}{n_k}\right)^{n_k/(\rho-\varepsilon)} \le |a_{n_k}|$ (otherwise the order would be less than ρ). This, in turn, gives

$$\ln\frac{1}{|a_{n_k}|} \le \frac{n_k}{\rho-\varepsilon}(\ln n_k - \ln e(\rho-\varepsilon)) = \frac{n_k \ln n_k}{\rho-\varepsilon}(1+o(1)), \quad k \to \infty.$$

Hence $A \ge \rho$ and formula (2.10) is proved. Formula (2.11) can be proved analogously. □

Remark 2.3. From the proof of Theorem 2.3 it follows that formula (2.10) can be used for functions of infinite order $\rho = \infty$ and formula (2.11) can be used for functions of maximal type $\sigma = \infty$ (if $0 < \rho < \infty$).

Remark 2.4. The limits in (2.10) and (2.11) will not change, if the coefficients a_n are replaced by $(n+1)a_n$. Hence, after differentiating, the order and the type of an entire function do not change.

Using Theorem 2.3 one can easily construct entire functions of arbitrary order and type. Let us give several examples.

Example 2.4. Consider the function

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n = \left(\frac{C}{n}\right)^{n/\alpha}, \quad C - const,$$

where $0 < \alpha < \infty$. By (2.10) we calculate

$$\rho = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln \dfrac{1}{\sqrt[n]{|a_n|}}} = \alpha.$$

Thus, the function $f(z)$ is entire of order $\rho = \alpha$.

Example 2.5. Consider the function

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_n = \left(\frac{e\alpha\beta}{n}\right)^{n/\alpha},$$

where $0 < \alpha < \infty$, $0 < \beta < \infty$. According to the previous example we have $\rho = \alpha$. By (2.11) we calculate

$$(\sigma e \rho)^{1/\rho} = \overline{\lim_{n\to\infty}}\, n^{1/\alpha} \sqrt[n]{|a_n|} = (\beta e \rho)^{1/\rho}.$$

Thus, the function $f(z)$ is entire of order $\rho = \alpha$ and of type $\sigma = \beta$.

Example 2.6. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |a_n| = \frac{1}{n^{n/\alpha_n}}, \quad \alpha_n > 0.$$

Then

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\ln \dfrac{1}{\sqrt[n]{|a_n|}}} = \overline{\lim_{n\to\infty}}\, \alpha_n.$$

Thus, if $\alpha := \overline{\lim_{n\to\infty}} \alpha_n < \infty$, then the function $f(z)$ is entire of order $\rho = \alpha$. In particular, for $\alpha = 0$ (e.g., $\alpha_n = 1/n$) we get an entire function of zero-order. If $\alpha = \infty$ and $\alpha_n = o(\ln n)$ (e.g., $\alpha_n = \ln^\delta n,\ \delta \in (0,1)$), then according to (1.2) the function $f(z)$ is entire of infinite order.

Example 2.7. The function $f(z) = \sum_{n=1}^{\infty} a_n z^n,\quad a_n = \left(\dfrac{\ln n}{n}\right)^{n/\rho}$ is entire of order ρ and maximal type.

Example 2.8. The function $f(z) = \sum_{n=2}^{\infty} a_n z^n,\quad a_n = \left(\dfrac{1}{n\ln n}\right)^{n/\rho}$ is entire of order ρ and minimal type.

Example 2.9. The function $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^n}{(2n+1)!}$ is entire of order $\rho = 1/2$ and type $\sigma = 1$.

3. Zeros of an entire function

3.1. Entire functions with a finite number of zeros. An entire function $f(z) \not\equiv 0$ can either have no zeros or a finite number of zeros or a countable set of zeros. The function

$$f(z) = e^{g(z)}, \tag{3.1}$$

where $g(z)$ is an entire function, is entire and has no zeros. Conversely, if the entire function $f(z)$ has no zeros, then the function $F(z) := \ln f(z)$ (where we take one of the regular branches of the logarithm) is entire, and $F'(z) = \frac{f'(z)}{f(z)}$. Hence, $f(z)$ has the form (3.1), where $g(z) = F(z)$ is an entire function.

If the entire function $f(z)$ has a finite number of zeros $z_1, z_2, \ldots, z_n$ (counting multiplicity), then the function $\Phi(z) = \frac{f(z)}{\varphi(z)}$, where $\varphi(z) = \prod_{k=1}^{n}(z - z_k)$, has no zeros, and consequently, it can be represented in the form (3.1). Hence

$$f(z) = e^{g(z)} \prod_{k=1}^{n} (z - z_k). \tag{3.2}$$

Thus, entire functions of the form (3.2) (and only they) have a finite number of zeros.

Formula (3.2) can be generalized to the case when an entire function has an infinite number of zeros. This was made by Weierstrass and Hadamard. In this case the finite product is replaced by an infinite one, in which it is necessary in general to add certain multipliers in order to make the infinite product convergent. For convenience to the reader in the next section we provide some auxiliary information on infinite products.

3.2. Infinite products. The numeric infinite product $\prod_{n=1}^{\infty} g_n$ is called *convergent*, if for a certain N there exists $\lim_{k\to\infty} \prod_{n=N}^{k} g_n \neq 0, \infty$. For a convergent infinite product we have

$$\lim_{n\to\infty} g_n = \frac{\lim_{n\to\infty} \prod_{k=N}^{n} g_k}{\lim_{n\to\infty} \prod_{k=N}^{n-1} g_k} = 1.$$

Therefore, sometimes it is convenient to write the common factor in the form $g_n = 1 + v_n$. The following proposition establishes a connection between infinite products and series.

Proposition 3.1. *The infinite product* $\prod_{n=1}^{\infty} g_n$ *converges if and only if the series* $\sum_{n=N}^{\infty} \ln g_n$ *converges for certain* N.

Proof. If the series is convergent, then we have

$$\exp\Big(\sum_{n=N}^{\infty} \ln g_n\Big) = \prod_{n=N}^{\infty} g_n \tag{3.3}$$

and hence the infinite product is convergent too. Conversely, we have

$$\ln \prod_{n=N}^{k} g_n = \sum_{n=N}^{k} \ln g_n + 2\pi\mu_k i, \quad \mu_k \in \mathbb{Z}.$$

Thus, if the product is convergent, then $\ln g_n \to 0$ and μ_k stabilizes to a certain $\mu \in \mathbb{Z}$. Consequently, the series is convergent too. $\square$

The infinite product $\prod_{n=1}^{\infty} g_n$ is called *absolutely convergent*, if for a certain N the series $\sum_{n=N}^{\infty} |\ln g_n|$ converges. According to (3.3) in an absolutely convergent infinite product one can arbitrarily change the order of factors both without loss of the convergence and without change of the value of the product.

Proposition 3.2. *The infinite product* $\prod_{n=1}^{\infty}(1 + v_n)$ *is absolutely convergent if and only if the series* $\sum_{n=1}^{\infty} |v_n|$ *converges.*

Proof. Let $|v_n| \le 1/2$. Since

$$\ln(1 + v_n) = v_n \sum_{j=0}^{\infty} (-1)^j \frac{v_n^j}{j+1},$$

we have the following two-way estimate:

$$\frac{1}{2}|v_n| = |v_n|\Big(1 - \frac{1}{2^2} - \frac{1}{2^3} - \dots\Big) \le |\ln(1 + v_n)| \le |v_n|\Big(1 + \frac{1}{2^2} + \frac{1}{2^3} + \dots\Big) = \frac{3}{2}|v_n|,$$

which proves this proposition. $\square$

Let the functions $g_n(z)$, $n \in \mathbb{N}$, be defined on a set S. The functional infinite product $\prod_{n=1}^{\infty} g_n(z)$ is called *uniformly convergent* on S, if for a certain N the sequence $\prod_{n=N}^{k} g_n(z)$ converges uniformly as $k \to \infty$ to some function $h_N(z)$, defined on S. Obviously, in this case $h_N(z) \ne 0$ for all $z \in S$.

According to the Weierstrass theorem on the limit of a uniformly convergent sequence of analytic functions (see, e.g., [2]), we get that if the functions $g_n(z)$ are analytic in a domain D and the infinite product $\prod_{n=1}^{\infty} g_n(z)$ is uniformly convergent on D, then this product is an analytic function on D.

Proposition 3.3. *If for certain* N *the series* $\sum_{n=N}^{\infty} \ln g_n(z)$ *is uniformly convergent on* S, *then so is also the infinite product* $\prod_{n=1}^{\infty} g_n(z)$.

Proof. We can choose N so that the function

$$h_N(z) := \exp\Big(\sum_{n=N}^{\infty} \ln g_n(z)\Big)$$

is bounded on S. Further, we have the estimate

$$\Big|h_N(z)-\prod_{n=N}^{k}g_n(z)\Big| = \Big|\exp\Big(\sum_{n=N}^{\infty}\ln g_n(z)\Big)-\exp\Big(\sum_{n=N}^{k}\ln g_n(z)\Big)\Big|$$

$$=|h_N(z)|\Big|1-\exp\Big(-\sum_{n=k+1}^{\infty}\ln g_n(z)\Big)\Big|\le C\Big|\sum_{n=k+1}^{\infty}\ln g_n(z)\Big|,$$

which proves the proposition. □

3.3. Expansion of an entire function into an infinite product. Let the entire function $f(z)$ have infinitely many zeros. Since in each circle $|z|<R$, $f(z)$ has only a finite number of zeros, we can enumerate them (counting multiplicity) as follows

$$\underbrace{0,0,\dots,0}_{m},z_1,z_2,\dots,z_n,\dots \qquad 0<|z_n|\le|z_{n+1}|,\quad \lim_{n\to\infty}|z_n|=\infty, \tag{3.4}$$

where $z=0$ is a zero of multiplicity m (if $f(0)\ne 0$, we put $m=0$). We agree below that the sequence of zeros of an entire function is always enumerated in this way. In this case it is called the ordered sequence of zeros. The following theorem allows one to construct entire functions with arbitrary a priori given zeros.

Theorem 3.1 (Weierstrass). *For each sequence (3.4) of complex numbers, there exists an entire function $f(z)$ such that its zeros coincide with this sequence.*

Proof. Consider the infinite product

$$z^m\prod_{n=1}^{\infty}\Big(1-\frac{z}{z_n}\Big)\exp(P_n(z)),\quad\text{where}\ \ P_n(z)=\sum_{j=1}^{n}\frac{z^j}{jz_n^j}=\frac{z}{z_n}+\dots+\frac{z^n}{nz_n^n}.$$

It is sufficient to show that it converges uniformly in each circle $D_R:=\{|z|<R\}$ to a function $\varphi(z)$, which satisfies the conditions of Theorem 3.1. According to Proposition 3.3 one should check if for certain N the series

$$\sum_{n=N}^{\infty}\Big(\ln\Big(1-\frac{z}{z_n}\Big)+P_n(z)\Big) \tag{3.5}$$

is uniformly convergent in D_R. Fix R and let $N=N(R)$ be such that

$$|z_n|\ge 2R\ \text{ for }\ n\ge N.$$

Then for $z\in D_R$ we have $\Big|\dfrac{z}{z_n}\Big|<\dfrac{1}{2}$, $n\ge N$. Hence, $\ln\Big(1-\dfrac{z}{z_n}\Big)$ can be expanded into the power series

$$\ln\Big(1-\frac{z}{z_n}\Big)=-\sum_{j=1}^{\infty}\frac{z^j}{jz_n^j},$$

and we obtain the estimate

$$\Big|\ln\Big(1-\frac{z}{z_n}\Big)+P_n(z)\Big|=\Big|\sum_{j=n+1}^{\infty}\frac{z^j}{jz_n^j}\Big|<\sum_{j=n+1}^{\infty}\frac{1}{2^j}=\frac{1}{2^n},\quad z\in D_R,$$

which implies the uniform convergence of the series (3.5) in the circle D_R. Thus, the function

$$\varphi(z) = z^m \prod_{n=1}^{\infty} \Big(1 - \frac{z}{z_n}\Big) \exp\Big(\frac{z}{z_n} + \ldots + \frac{z^n}{n z_n^n}\Big)$$

satisfies the conditions of the theorem. Theorem 3.1 is proved. $\square$

Corollary 3.1. *For each sequence (3.4) and for each entire function $g(z)$, the function*

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} \Big(1 - \frac{z}{z_n}\Big) \exp\Big(\frac{z}{z_n} + \ldots + \frac{z^n}{n z_n^n}\Big) \tag{3.6}$$

is entire, and its zeros coincide with the sequence (3.4).

The inverse assertion is also valid:

Theorem 3.2. *Let $f(z)$ be an entire function with zeros (3.4). Then $f(z)$ has the form (3.6), where $g(z)$ is an entire function.*

Proof. According to the proof of Theorem 3.1 the zeros of the entire function $\varphi(z)$ coincide with the zeros of $f(z)$. Then $\frac{f(z)}{\varphi(z)}$ is an entire function without zeros. Consequently, $\frac{f(z)}{\varphi(z)} = e^{g(z)}$, where $g(z)$ is an entire function, and we arrive at (3.6). $\square$

3.4. Jensen's formula. The simplest characteristic of zeros of an entire function $f(z)$ is the counting function $n(r)$, which is equal to the number of zeros of $f(z)$ in the circle $|z| < r$ (each zero is counted with its multiplicity). In order to estimate $\ln M_f(r)$ from below using $n(r)$, we need Jensen's formula.

Theorem 3.3. *Let the function $f(z)$ be analytic in the circle $|z| < R$, $f(0) \neq 0$ and let $\{z_n\}$ be the ordered sequence of zeros of $f(z)$. Then for $0 < r < R$ the following formula holds*

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\varphi})| \, d\varphi = \ln |f(0)| + \sum_{|z_n| < r} \ln \frac{r}{|z_n|}; \tag{3.7}$$

equality (3.7) is called Jensen's formula.

Proof. Let $r \neq |z_n|$ for all n. Consider the function

$$F(z) = f(z) \prod_{|z_n| < r} \frac{r^2 - z\bar{z}_n}{r(z - z_n)}.$$

The function $F(z)$ (more precisely: its continuous continuation) has no zeros in the circle $|z| \leq r$, hence the function $\ln\, F(z)$ is analytic in this circle, and, consequently, the function $\ln |F(z)| = \mathrm{Re} \ln F(z)$ is harmonic. Using the mean-value formula for harmonic functions we get

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |F(re^{i\varphi})| \, d\varphi = \ln |F(0)|. \tag{3.8}$$

Obviously,

$$\ln |F(0)| = \ln |f(0)| + \sum_{|z_n| < r} \ln \frac{r}{|z_n|}.$$

For $z = re^{i\varphi}$ we calculate

$$\Big|\frac{r^2 - re^{i\varphi}\bar{z}_n}{r(re^{i\varphi} - z_n)}\Big| = \Big|\frac{r - e^{i\varphi}\bar{z}_n}{r - e^{-i\varphi}z_n}\Big| = 1,$$

Thus, in (3.8) we can replace $|F(re^{i\varphi})|$ by $|f(re^{i\varphi})|$ and we arrive at (3.7). It remains to notice that the functions on both sides in (3.7) are continuous functions of r for $0 < r < R$, since the function under integration has only a finite number of integrable singularities. Therefore, Jensen's formula is valid also for $r = |z_n|$ for all n, i.e. it holds for $0 < r < R$. Theorem 3.3 is proved. □

One can rewrite (3.7) also in another form. For this purpose we calculate the integral

$$J = \int_0^r \frac{n(t)}{t}\,dt.$$

Since $n(r) = 0$ for $r \le |z_1|$, this integral exists, and we obtain

$$J = \sum_{k=1}^{n(r)-1} \int_{|z_k|}^{|z_{k+1}|} \frac{n(t)}{t}\,dt + \int_{|z_{n(r)}|}^{r} \frac{n(t)}{t}\,dt = \sum_{k=1}^{n(r)-1} \int_{|z_k|}^{|z_{k+1}|} \frac{k}{t}\,dt + \int_{|z_{n(r)}|}^{r} \frac{n(r)}{t}\,dt.$$

Consequently,

$$J = \sum_{k=1}^{n(r)-1} \ln\Big(\frac{|z_{k+1}|}{|z_k|}\Big)^k + \ln\Big(\frac{r}{|z_{n(r)}|}\Big)^{n(r)}$$

$$= \ln\frac{|z_2|\cdot|z_3|^2\ldots|z_{n(r)}|^{n(r)-1}\cdot r^{n(r)}}{|z_1|\cdot|z_2|^2\ldots|z_{n(r)-1}|^{n(r)-1}\cdot|z_{n(r)}|^{n(r)}} = \sum_{|z_n|\le r} \ln\frac{r}{|z_n|}.$$

Therefore, Jensen's formula can be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\varphi})|d\varphi = \ln|f(0)| + \int_0^r \frac{n(t)}{t}\,dt. \tag{3.9}$$

It follows from (3.9) that

$$\int_0^r \frac{n(t)}{t}\,dt \le \ln M_f(r) - \ln|f(0)|,$$

hence

$$\int_0^r \frac{n(t)}{t}\,dt \le \ln\frac{M_f(r)}{|f(0)|}. \tag{3.10}$$

Inequality (3.10) connects the number of zeros of $f(z)$ in the circle $|z| < r$ with $M_f(r)$, i.e. with the maximum of the modulus of f. We call (3.10) *Jensen's inequality.*

3.5. The convergence exponent. Consider the sequence of complex numbers

$$a_1, a_2, \ldots, a_n, \ldots, \quad 0 < |a_n| \le |a_{n+1}|, \quad \lim|a_n| = \infty. \tag{3.11}$$

If the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda} \tag{3.12}$$

converges for a certain $\lambda > 0$, we shall say that the sequence (3.11) has a finite convergence exponent. The greatest lower bound of those λ, for which (3.12) converges, is called the *convergence exponent* of the sequence (3.11) and is denoted by τ : $\tau = \inf \lambda$. If (3.12) does not converges for any $\lambda > 0$, we put $\tau = \infty$.

Theorem 3.4. *The convergence exponent τ of the sequence (3.11) is calculated by the formula*

$$\tau = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln |a_n|}.$$

Proof. 1) Let $\alpha := \overline{\lim\limits_{n\to\infty}} \dfrac{\ln n}{\ln |a_n|} \neq \infty$. Then $\dfrac{\ln n}{\ln |a_n|} < \alpha + \varepsilon$, $n > N(\varepsilon)$. Consequently,

$$\frac{1}{|a_n|^\lambda} < \left(\frac{1}{n}\right)^{\frac{\lambda}{\alpha+\varepsilon}}, \quad n > N(\varepsilon).$$

Therefore, the series (3.12) converges for $\lambda > \alpha+\varepsilon$. By virtue of the arbitrariness of $\varepsilon > 0$, the series converges for any $\lambda > \alpha$, hence, $\tau \le \alpha$.

2) Suppose now that τ is finite. Then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\tau+\varepsilon}}$ converges for any $\varepsilon > 0$. Since

$$\lim_{n\to\infty} \frac{n}{|a_n|^{\tau+\varepsilon}} = 0,$$

we have for sufficiently large n : $\dfrac{\ln n}{\ln |a_n|} \le \tau + \varepsilon$, For $n \to \infty$ this yields $\alpha \le \tau + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\alpha \le \tau$. Comparing this inequality with the inequality $\alpha \ge \tau$ (see above), we conclude that $\tau = \alpha$. Thus we have proved that τ and α are finite simultaneously. Hence, they are infinite simultaneously too. Theorem 3.4 is proved. $\square$

Example 3.1. The sequence $\{e^n\}$ has the convergence exponent $\tau = 0$, since

$$\lim_{n\to\infty} \frac{\ln n}{\ln |a_n|} = \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Example 3.2. The sequence $\{n^{1/\beta}\}, \beta > 0$ has the convergence exponent $\tau = \beta$.

Example 3.3. The sequence $\{\ln n\}$ has the convergence exponent $\tau = \infty$.

3.6. The connection between the growth of an entire function and its zeros. It turns out that there is a close relation between the growth rate of the zeros of an entire function f and the growth of its maximum modulus $M_f(r)$.

Let us estimate the left-hand side of (3.10) from below. Let $\theta \in (0,1)$. Then we have

$$\int_0^r \frac{n(t)}{t} dt \ge \int_{\theta r}^r \frac{n(t)}{t}\, dt \ge n(\theta r) \int_{\theta r}^r \frac{dt}{t} = n(\theta r) \ln \frac{1}{\theta}.$$

Hence, by virtue of (3.10),

$$n(\theta r) \le \frac{1}{\ln \dfrac{1}{\theta}} \ln \frac{M_f(r)}{|f(0)|}\,. \tag{3.13}$$

Let $f(z)$ be an entire function of the finite order ρ. Then $\ln M_f(r) < r^{\rho+\varepsilon}$ for $r > R_0(\varepsilon)$. It follows from (3.13) that

$$\frac{n(\theta r)}{r^{\rho+\varepsilon}} < \frac{1}{\ln\dfrac{1}{\theta}} - \frac{\ln|f(0)|}{r^{\rho+\varepsilon}\ln\dfrac{1}{\theta}}, \quad r > R_0(\varepsilon).$$

Therefore,

$$\overline{\lim_{r\to\infty}}\frac{n(\theta r)}{r^{\rho+\varepsilon}} \le \frac{1}{\ln\dfrac{1}{\theta}}.$$

Replacing θr in the preceding inequality by r we obtain

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{r^{\rho+\varepsilon}} \le \frac{1}{\theta^{\rho+\varepsilon}\ln\dfrac{1}{\theta}},$$

where θ is an arbitrary number from $(0,1)$. Take $\theta = \exp(-\frac{1}{\rho+\varepsilon})$; for this θ the right-hand side of the inequality attains its minimum. Hence,

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{r^{\rho+\varepsilon}} \le e(\rho+\varepsilon). \tag{3.14}$$

Further, let $f(z)$ be an entire function of order ρ $(0<\rho<\infty)$ and of finite type σ. Then $\ln M_f(r) < (\sigma+\varepsilon)r^\rho$ for $r > R_1(\varepsilon)$. It follows from (3.13) that

$$\frac{n(\theta r)}{r^{\rho}} < \frac{(\sigma+\varepsilon)}{\ln\dfrac{1}{\theta}} - \frac{\ln|f(0)|}{r^{\rho}\ln\dfrac{1}{\theta}}, \quad r > R_1(\varepsilon),$$

and consequently,

$$\overline{\lim_{r\to\infty}}\frac{n(\theta r)}{r^{\rho}} \le \frac{(\sigma+\varepsilon)}{\ln\dfrac{1}{\theta}}.$$

Replacing again θr by r, we calculate

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{r^{\rho}} \le \frac{\sigma+\varepsilon}{\theta^{\rho}\ln\dfrac{1}{\theta}}.$$

Taking here $\theta = e^{-1/\rho}$ and using the arbitrariness of $\varepsilon > 0$, we obtain finally

$$\overline{\lim_{r\to\infty}}\frac{n(r)}{r^{\rho}} \le \sigma e\rho. \tag{3.15}$$

Let $f(z)$ have an infinite set of zeros. Since $n(|z_n|+1) \ge n$, in (3.14) and (3.15) we can replace $n(r)$ with n, and r with $|z_n|$. Thus, the following theorem has been proved:

Theorem 3.5. *Let $f(z)$ be an entire function of a finite order ρ, and let $\{z_n\}$ be the ordered sequence of its zeros. Then we have:*

$$\overline{\lim_{n\to\infty}}\frac{n}{|z_n|^{\rho+\varepsilon}} \le e(\rho+\varepsilon), \quad \varepsilon > 0. \tag{3.16}$$

If in addition $0 < \rho < \infty$ *and if* $f(z)$ *is of finite type* σ, *then*

$$\overline{\lim_{n\to\infty}} \frac{n}{|z_n|^\rho} \le \sigma e \rho. \tag{3.17}$$

Using Theorem 3.5 we can prove the following assertion, which is due to Hadamard.

Theorem 3.6. *Let* $f(z)$ *be the entire function of a finite order* ρ. *Then the convergence exponent* τ *of the sequence of its zeros is finite, and* $\tau \le \rho$.

Proof. It follows from (3.16) that

$$\text{or} \quad \frac{1}{|z_n|^\lambda} < \Big(\frac{e(\rho+2\varepsilon)}{n}\Big)^{\frac{\lambda}{\rho+\varepsilon}}, \quad n > N_1(\varepsilon).$$

Hence, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{|z_n|^\lambda}$ converges for $\lambda > \rho+\varepsilon$. This yields the assertion of Theorem 3.6. □

In particular, from Theorems 3.5, 3.6 we derive the following uniqueness theorems.

Theorem 3.7. *Let the order of the entire function* $f(z)$ *be not greater than* ρ. *If* $f(z_n) = 0$, $n \ge 1$, *and the convergence exponent of the sequence* $\{z_n\}$ *is greater than* ρ, *then* $f(z) \equiv 0$.

Theorem 3.8. *Suppose that the type of the entire function* $f(z)$, *with respect to the order* ρ, *is not greater than* σ. *If* $f(z_n) = 0$, $n \ge 1$, *and* $\displaystyle\overline{\lim_{n\to\infty}} \frac{n}{|z_n|^\rho} > \sigma e \rho$, *then* $f(z) \equiv 0$.

3.7. Hadamard's factorization theorem and Borel's theorem. First we prove two auxiliary assertions.

Lemma 3.1. *Let a sequence* $\{z_n\}$ *of nonzero numbers be given. If there is integer* p, *such that*

$$\sum_{n=1}^\infty \frac{1}{|z_n|^{p+1}} < \infty, \tag{3.18}$$

then the function

$$f(z) = \prod_{n=1}^\infty \Big(1 - \frac{z}{z_n}\Big) \exp(p_n(z)), \quad p_n(z) := \sum_{j=1}^p \frac{z^j}{j z_n^j} = \frac{z}{z_n} + \cdots + \frac{z^p}{p z_n^p} \tag{3.19}$$

(if $p = 0$ *we put* $p_n(z) \equiv 0$), *is entire, and its zeros (counting multiplicities) coincide with the sequence* $\{z_n\}$.

Proof. The proof is similar to that of Theorem 3.1. Fix arbitrary $R > 0$. It is sufficient to check if for certain N the series

$$\sum_{n=N}^\infty \Big(\ln\Big(1 - \frac{z}{z_n}\Big) + p_n(z)\Big) \tag{3.20}$$

is uniformly convergent in $D_R = \{z : |z| < R\}$. Choose $N = N(R)$ such that

$$|z_n| \ge 2R \ \text{ for } \ n \ge N.$$

Then $\left|\dfrac{z}{z_n}\right| < \dfrac{1}{2}$ for $n \ge N$, $z \in D_R$. Hence, we get the estimate

$$\Big|\ln\Big(1-\frac{z}{z_n}\Big)+p_n(z)\Big| = \Big|\sum_{j=p+1}^{\infty}\frac{z^j}{j z_n^j}\Big| \le \frac{|z|^{p+1}}{|z_n|^{p+1}}\sum_{j=0}^{\infty}\frac{|z|^j}{|z_n|^j} < \frac{R^{p+1}}{|z_n|^{p+1}}\sum_{j=0}^{\infty}\frac{1}{2^j} = \frac{2R^{p+1}}{|z_n|^{p+1}},$$

which implies the uniform convergence of the series (3.20) in the circle D_R.

Lemma 3.2. *Let the function* $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n$ *be analytic in the circle* $|z|<R$, *and* $\operatorname{Re} f(z) \le \alpha$ *for* $|z|<R$. *Then*

$$|a_n| \le \frac{2(\alpha - \operatorname{Re} f(0))}{R^n}, \quad n \ge 1. \tag{3.21}$$

Proof. Let

$$\Phi(z) = \alpha - f(z) = \alpha - \sum_{n=0}^{\infty} a_n z^n =: \sum_{n=0}^{\infty} b_n z^n, \quad |z|<R.$$

Thus, $\operatorname{Re}\Phi(z) \ge 0$ for $|z|<R$ and $|b_n| = |a_n|$ for $n \ge 1$. For $r \in (0,R)$ we have

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{\Phi(z)}{z^{n+1}}\,dz = \begin{cases} b_n, & n \ge 0,\\ 0, & n<0.\end{cases} \tag{3.22}$$

On the other hand we have

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{\Phi(z)}{z^{n+1}}\,dz = \frac{1}{2\pi r^n}\int_0^{2\pi}\Phi(z)e^{-in\varphi}\,d\varphi, \quad z = re^{i\varphi}.$$

Combining this with (3.22) we get

$$b_n r^n = \frac{1}{2\pi}\int_0^{2\pi}\Phi(z)e^{-in\varphi}\,d\varphi, \quad 0 = \frac{1}{2\pi}\int_0^{2\pi}\overline{\Phi(z)}e^{-in\varphi}\,d\varphi, \quad n \ge 1.$$

Summing up these two equalities term by term, we get

$$b_n = \frac{1}{\pi r^n}\int_0^{2\pi}\operatorname{Re}\Phi(z)e^{-in\varphi}\,d\varphi, \quad n \ge 1,$$

and hence, according to the mean-value theorem,

$$|a_n| = |b_n| \le \frac{1}{\pi r^n}\int_0^{2\pi}\operatorname{Re}\Phi(z)\,d\varphi = \frac{2\operatorname{Re}\Phi(0)}{r^n} = \frac{2(\alpha-\operatorname{Re} f(0))}{r^n}, \quad n \ge 1.$$

Taking the limit as $r \to R$ we get (3.21). Lemma 3.2 is proved. □

Theorem 3.9 (Hadamard's factorization theorem). *Let* $f(z)$ *be an entire function of a finite order* $\rho < \infty$; *let* $z_1, z_2, \ldots$ $(z_k \ne 0)$ *be the ordered sequence of its zeros,* *and let* $z = 0$ *be a zero of* $f(z)$ *of multiplicity* $m \geq 0$. *Let* $p \geq 0$ *be the smallest integer such that*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{p+1}} < \infty. \tag{3.23}$$

Then

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} \Big(1 - \frac{z}{z_k}\Big) \exp(p_k(z)), \quad p_k(z) = \sum_{j=1}^{p} \frac{z^j}{j {z_k}^j}, \tag{3.24}$$

where $g(z)$ *is a polynomial of degree* $n \leq \rho$.

Proof. By Theorem 3.6, the sequence $\{z_k\}$ has a finite convergence exponent $\tau \leq \rho$. Therefore, there exists the smallest integer $p \geq 0$ such that (3.23) is fulfilled. Clearly, $p \leq \tau \leq p+1$. According to Lemma 3.1, the entire function

$$F(z) = z^m \prod_{k=1}^{\infty} \Big(1 - \frac{z}{z_k}\Big) \exp(p_k(z)) \tag{3.25}$$

has the same zeros as $f(z)$. Consequently, $\varphi(z) = f(z)/F(z)$ is an entire function without zeros. Hence, $\varphi(z) = e^{g(z)}$, where $g(z)$ is an entire function. This yields $f(z) = e^{g(z)} F(z)$, i.e. (3.24) is valid. It remains to show that $g(z)$ is a polynomial of degree $n \leq \rho$.

For an arbitrary R we rewrite $f(z)$ as follows

$$f(z) = z^m \prod_{k=1}^{N} \Big(1 - \frac{z}{z_k}\Big) \cdot \Psi_N(z), \quad \Psi_N(z) = \exp\Big(g(z) + \sum_{k=1}^{N} p_k(z)\Big) \prod_{k=N+1}^{\infty} \Big(1 - \frac{z}{z_k}\Big) \exp(p_k(z)),$$

where $N = N(R)$ is such that $|z_k| \leq R$ for $k \leq N$ and $|z_k| > R$ for $k > N$. Then for $R \geq 1/2$ one has

$$M_f(2R) = \max_{|z|=2R} |f(z)| \geq (2R)^m \prod_{k=1}^{N} \Big(\frac{2R}{R} - 1\Big) \cdot \max_{|z|=2R} |\Psi_N(z)| \geq \max_{|z|=2R} |\Psi_N(z)|.$$

Since $\Psi_N(z)$ is an entire function, we infer from the maximum modulus principle that

$$|\Psi_N(z)| < M_f(2R), \quad |z| < 2R. \tag{3.26}$$

In the circle $|z| < R$ the function $\Psi_N(z)$ can be represented in the form $\Psi_N(z) = e^{g_N(z)}$, where $g_N(z) = g(z) + \sum_{k=1}^{N} p_k(z) + \sum_{k=N+1}^{\infty} \Big(\ln\Big(1 - \frac{z}{z_k}\Big) + p_k(z)\Big)$. The function $g_N(z)$ is analytic in the circle $|z| < R$. Consider the Taylor coefficients $C_n = \frac{g_N^{(n)}(0)}{n!}$ of $g_N(z)$. We have

$$C_n = \frac{g^{(n)}(0)}{n!} - \sum_{k=N+1}^{\infty} \frac{1}{n z_k^n}, \quad n > p, \tag{3.27}$$

since $p_k(z)$ is a polynomial of degree p. It follows from (3.26) that $\operatorname{Re} g_N(z) < \ln M_f(2R)$ for $|z| < R$. By virtue of Lemma 3.2 this yields

$$|C_n| \leq 2 \frac{\ln M_f(2R) - \operatorname{Re} C_0}{R^n}, \quad n \geq 1.$$

Since $\rho \geq \tau \geq p$, in view of (3.27) we have

$$\Big|\frac{g^{(n)}(0)}{n!}\Big| < 2\frac{\ln M_f(2R) - \operatorname{Re} C_0}{R^n} + \sum_{k=N+1}^{\infty} \frac{1}{n|z_k|^n}, \quad n > \rho, \tag{3.28}$$

where the series converges, because $n > \tau$. Hence, $\lim\limits_{R\to\infty} \sum\limits_{k=N(R)+1}^{\infty} \dfrac{1}{|z_k|^n} = 0$ for $n > \rho$. Since $\ln M_f(2R) < (2R)^{\rho+\varepsilon}$ for sufficiently large R, one has

$$\lim_{R\to\infty} \frac{\ln M_f(2R) - \operatorname{Re} g(0)}{R^n} = 0, \quad n > \rho.$$

Taking the limit in (3.28) as $R \to \infty$, we obtain $g^{(n)}(0) = 0$ for $n > \rho$. Thus, $g(z)$ is a polynomial of degree $n \leq \rho$, and the theorem is proved. $\square$

A function $F(z)$ of the form (3.25) is called a *canonical product*, and the number p is called the *genus* of this canonical product.

Theorem 3.10 (Borel). *Let the entire function $f(z)$ have the form (3.24), where the sequence $\{z_k\}$ has the convergence exponent τ, p is the smallest integer satisfying (3.23), and $g(z)$ is a polynomial of degree n. Then $f(z)$ has finite order ρ, and*

$$\rho = \max\{\tau, n\}.$$

Moreover, if $\tau < n$ or if the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{|z_k|^\tau}$ converges, then $f(z)$ is of finite type.

Proof. First we estimate the modulus of each term of the canonical product. Consider the function

$$\varphi(u) = (1-u)\exp\Big(u + \frac{u^2}{2} + \ldots + \frac{u^p}{p}\Big).$$

For $|u| \leq 1/2$ we have

$$|\varphi(u)| = \Big|\exp\Big(\ln(1-u) + u + \frac{u^2}{2} + \ldots + \frac{u^p}{p}\Big)\Big|$$

$$= \Big|\exp\Big(-\sum_{k=p+1}^{\infty} \frac{u^k}{k}\Big)\Big| \leq \exp(2|u|^{p+1}) \leq \exp(2|u|^\lambda),$$

where $\lambda \leq p+1$. For $|u| > 1/2$ we calculate

$$|\varphi(u)| < (1+|u|)\exp(|u| + |u|^2 + \ldots + |u|^p)$$

$$= \exp\Big(\ln(1+|u|) + |u|^p\Big(1 + \frac{1}{|u|} + \ldots + \frac{1}{|u|^{p-1}}\Big)\Big) \leq \exp(C|u|^p) \leq \exp(C2^\lambda|u|^\lambda),$$

where $\lambda \geq p$. Therefore, for each u we have

$$|\varphi(u)| \leq \exp(C|u|^\lambda), \quad p \leq \lambda \leq p+1. \tag{3.29}$$

Choose λ such that the series $\sum_{k=1}^{\infty}\frac{1}{|z_k|^{\lambda}}$ converges. If the series $\sum_{k=1}^{\infty}\frac{1}{|z_k|^{\tau}}$ converges, then $p < \tau \le p+1$, and we put $\lambda = \tau$; if this series does not converge, then $p \le \tau < p+1$, and we can take $\lambda = \tau + \varepsilon$, where $0 < \varepsilon \le p + 1 - \tau$. Taking $u = z/z_k$ in (3.29) we obtain

$$\Big|\Big(1 - \frac{z}{z_k}\Big)\exp(p_k(z))\Big| \le \exp\Big(C\frac{|z|^{\lambda}}{|z_k|^{\lambda}}\Big).$$

We note that if $g(z) = B_0 + \ldots + B_n z^n$, then for each $\varepsilon > 0$,

$$|z^m e^{g(z)}| < \exp((|B_n| + \varepsilon)|z|^n) \ \text{ for } \ |z| > R_0(\varepsilon).$$

Consequently,

$$|f(z)| < \exp\Big((|B_n| + \varepsilon)|z|^n + C\sum_{k=1}^{\infty}\frac{1}{|z_k|^{\lambda}}|z|^{\lambda}\Big), \quad |z| > R_0(\varepsilon). \tag{3.30}$$

It follows from (3.30) that $f(z)$ has finite order ρ, and $\rho \le \max\{n, \lambda\}$. Since $\lambda = \tau$ or $\lambda = \tau + \varepsilon$, we obtain, by virtue of the arbitrariness of $\varepsilon > 0$, that $\rho \le \max\{n, \tau\}$. On the other hand, by Theorem 3.6 $\tau \le \rho$ and by Theorem 3.9 $n \le \rho$. Hence,

$$\rho = \max\{n, \tau\}.$$

Furthermore, if $n > \tau$, then $\rho = n$. Take $\tau < \lambda < n$, and according to (3.30) we get for sufficiently large $|z|$ that $|f(z)| < \exp(C_1|z|^n)$. From this we derive that the type of the function $f(z)$ is finite. If $n \le \tau$, and the series $\sum_{k=1}^{\infty}\frac{1}{|z_k|^{\tau}}$ converges, one can put $\lambda = \tau$ in (3.30). Then $|f(z)| < \exp(C_2|z|^{\tau})$, i.e. the type of $f(z)$ is finite again. $\square$

Corollary 3.2. *Let $f(z)$ be an entire function of finite order ρ and let τ be the convergence exponent of its zeros. If ρ is not integer, then $\rho = \tau$.*

Corollary 3.3. *Each entire function $f(z)$ of order $\rho \in [0, 1)$ can be represented in the form*

$$f(z) = Cz^m \prod_{k=1}^{\infty}\Big(1 - \frac{z}{z_k}\Big)$$

(for a polynomial there is a finite number of factors), where C is constant.

Example 3.4. Let us expand the function $\sin z$ into an infinite product. The zeros of this function are 0, $\pm k\pi$, $k \in \mathbb{N}$, moreover $\rho = 1$, $\tau = 1$, $n \le 1$. Since

$$\Big(1 - \frac{z}{k\pi}\Big)\exp\Big(\frac{z}{k\pi}\Big)\Big(1 + \frac{z}{k\pi}\Big)\exp\Big(-\frac{z}{k\pi}\Big) = \Big(1 - \frac{z^2}{k^2\pi^2}\Big),$$

we have

$$\sin z = ze^{az+b}\prod_{k=1}^{\infty}\Big(1 - \frac{z^2}{k^2\pi^2}\Big).$$

Since $\lim_{z\to 0}\frac{\sin z}{z} = 1$, we calculate $e^b = 1$. Since $\sin(-z) = -\sin z$, we get $e^{az} = e^{-az}$, i.e. $a = 0$. Consequently,

$$\sin z = z\prod_{k=1}^{\infty}\Big(1 - \frac{z^2}{k^2\pi^2}\Big). \tag{3.31}$$

Analogously, one can calculate

$$\cos z = \prod_{k=1}^{\infty}\Big(1 - \frac{z^2}{(k-1/2)^2\pi^2}\Big).$$

4. A-points of an entire function

A point z such that $f(z) = A$ is called *A-point* of f.

Theorem 4.1. *Let $f(z)$ be an entire function of order ρ $(0 < \rho < \infty)$. If ρ is not integer, then the sequence of A-points has the convergence exponent $\tau_A = \rho$. If ρ is integer, then the sequence of A-points has the convergence exponent $\tau_A = \rho$ for all A with the exception of at most one value of A.*

Proof. The function $\varphi(z) = f(z) - A$ has the order ρ. If ρ is not an integer, then by Corollary 3.2, $\tau_A = \rho$, and the first assertion of the theorem is proved. Let now ρ be an integer, and suppose that $\tau_A < \rho$ and $\tau_B < \rho$ for some complex numbers A and B $(B \neq A)$. Then

$$f(z) - A = e^{P(z)}\varphi(z), \quad f(z) - B = e^{Q(z)}\psi(z),$$

where $P(z)$ and $Q(z)$ are polynomials of degree ρ, and $\varphi(z)$ and $\psi(z)$ are entire functions of order less than ρ. Then

$$A - B = e^{Q(z)}\psi(z) - e^{P(z)}\varphi(z). \tag{4.1}$$

Differentiating (4.1), we get

$$0 = e^{Q(z)}[Q'(z)\psi(z) + \psi'(z)] - e^{P(z)}[P'(z)\varphi(z) + \varphi'(z)]. \tag{4.2}$$

The functions in the square brackets are not identically zero. Indeed, suppose, for example, that $Q'(z)\psi(z) + \psi'(z) \equiv 0$. Then $\psi(z) = \exp(-Q(z) + C)$, which contradicts the fact that the order of $\psi(z)$ is less than ρ.

It follows from (4.2) that

$$e^{Q(z)-P(z)}(Q'(z)\psi(z) + \psi'(z)) = P'(z)\varphi(z) + \varphi'(z). \tag{4.3}$$

Let $P(z) = a_0 z^\rho + \ldots + a_\rho$, $Q(z) = b_0 z^\rho + \ldots + b_\rho$. If $a_0 \neq b_0$, then the left-hand side in (4.3) is an entire function of order ρ, and the right-hand side is an entire function of order less than ρ, which is impossible. Hence, $a_0 = b_0$. It follows from (4.1) that

$$(A - B)e^{-a_0 z^\rho} = e^{Q_1(z)}\psi(z) - e^{P_1(z)}\varphi(z),$$

where $Q_1(z)$ and $P_1(z)$ are polynomials of degree less than ρ. Therefore, the left-hand side in this equality is an entire function of order ρ, and the right-hand side is an entire function of order less than ρ. This contradiction means that for different A and B the relations $\tau_A < \rho$ and $\tau_B < \rho$ cannot be fulfilled simultaneously. □

A is called *a Borel exceptional value* if $\tau_A \neq \rho$. Theorem 4.1 says that an entire function has at most one Borel exceptional value $A \in \mathbf{C}$, and this can happen only if its order is an integer.

Example 4.1. Let $f(z) = e^{z^2}\sin z$. The function $f(z)$ is entire of order $\rho = 2$. The exeptional value here is $A = 0$, for which $\tau_0 = 1$.

5. Phragmen-Lindelöf's theorem

The function $M_f(r)$ does not contain enough information for the full characterization of the behavior of $f(z)$ at infinity. Consider, for example, the function $f(z) = e^z$. For $z = re^{i\varphi}$ one has $|e^z| = e^{\mathrm{Re}\, z} = e^{r\cos\varphi}$, $M_f(r) = e^r$. For $\varphi \in (-\pi/2 + \varepsilon, \pi/2 - \varepsilon)$, $\varepsilon > 0$, we have $|e^z| > e^{r\sin\varepsilon}$, i.e. $|f(z)| \to \infty$ for $z \to \infty$. For $\varphi \in (\pi/2 + \varepsilon, 3\pi/2 - \varepsilon)$, $\varepsilon > 0$, we get $|e^z| < e^{-r\sin\varepsilon}$, i.e. $|f(z)| \to 0$ for $z \to \infty$.

The following theorem gives us the important information on behavior of an entire function inside a sector if its behavior on the boundaries of this sector is known.

Theorem 5.1 (Pragmen-Lindelöf). *Let $f(z)$ be an entire function of a finite order not greater than than ρ, and $\alpha < 1/\rho$. Let B be a sector of angle $\pi\alpha$ in the (z)-plane. If $|f(z)| \le C$ on the boundaries of B, then $|f(z)| \le C$ inside B as well with the same constant C.*

Proof. Without loss of generality one can take $B = \{z : |\arg z| < \pi\alpha/2\}$. Let β and γ be such that

$$\rho < \beta < \gamma < 1/\alpha. \tag{5.1}$$

Let us consider the function $F(z) := f(z)\exp(-\varepsilon z^\gamma)$, where $\varepsilon > 0$, and as an analytic branch of the function z^γ we choose its principal value. Let $z = re^{i\varphi} \in \overline{B}$, i.e. $|\varphi| \le \pi\alpha/2$. By virtue of (5.1), $\gamma|\varphi| \le \gamma\pi\alpha/2 < \pi/2$, and consequently, $\cos\gamma\varphi \ge a > 0$. Then

$$|\exp(-\varepsilon z^\gamma)| = \exp(-\varepsilon(\cos\gamma\varphi)r^\gamma) \le \exp(-\varepsilon a r^\gamma), \quad |\varphi| \le \pi\alpha/2.$$

Since $f(z)$ is an entire function of order not greater than ρ, we have for sufficiently large r : $|f(re^{i\varphi})| < \exp(r^\beta)$. This yields

$$|F(re^{i\varphi})| < \exp(r^\beta - a\varepsilon r^\gamma) \le C, \quad r > R_0(\varepsilon), \quad |\varphi| \le \pi\alpha/2. \tag{5.2}$$

Consider the compact set $S_\varepsilon = \{z = re^{i\varphi} : |\varphi| \le \pi\alpha/2,\ r \le R_0(\varepsilon)\}$. Then we have $|F(z)| \le C$ on ∂S_ε. According to the maximum modulus principle for analytic functions, $|F(z)| \le C$ is valid for all $z \in S_\varepsilon$. Together with (5.2) this yields $|F(z)| \le C$, $|\varphi| \le \pi\alpha/2$ or $|f(z)| \le C\exp(-\varepsilon a r^\gamma)$, $|\varphi| \le \pi\alpha/2$. By virtue of the arbitrariness of $\varepsilon > 0$, we obtain $|f(z)| \le C$, $z \in B$, and Theorem 5.1 is proved. □

Corollary 5.1. *If the entire function $f(z)$ of order $\rho < 1/2$ is bounded on a certain ray $\arg z = \varphi_0$, then $f(z) \equiv const.$*

This assertion is not valid for entire functions of order $\rho \ge 1/2$. For example, the function $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$ has the order $\rho = 1/2$ and is bounded for $z > 0$.

We mention that in the literature there exist further more general Phragmen-Lindelöf-type results for functions being analytic on simply connected regions G – we omit details.

6. Entire functions of exponential type

6.1. The Borel transform. An entire function is called an *entire function of exponential type* if its order is $\rho < 1$ or $\rho = 1$ with a finite type σ (i.e. the growth of the function is not greater than of the first order and normal type). Such functions are called also *entire function of finite degree*, and the value

$$\kappa = \overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r}$$

is called the *degree* of the entire function $f(z)$. According to (2.7), the degree of an entire function equals its type with respect to the order $\rho = 1$. If $\rho < 1$, then $\kappa = 0$, and if $\rho > 1$, then $\kappa = \infty$. For an entire function of exponential type in the power series (1.1) it is convenient to use the designation $b_n = n!a_n$, i.e. formula (1.1) takes the form

$$f(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n. \tag{6.1}$$

For $\rho = 1$, formula (2.11) yields

$$\kappa e = \overline{\lim_{n\to\infty}}\, n \sqrt[n]{\frac{|b_n|}{n!}}.$$

Since $n! \sim \sqrt{2\pi n}(n/e)^n$, we get

$$\kappa = \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|}. \tag{6.2}$$

Denote

$$\gamma(t) = \sum_{n=0}^{\infty} \frac{b_n}{t^{n+1}}. \tag{6.3}$$

By virtue of (6.2), the series (6.3) converges for $|t| > \kappa$, and on the circle $|t| = \kappa$ the function $\gamma(t)$ has at least one singular point. The function $\gamma(t)$ is called the *Borel transform* of $f(z)$.

We note that if $f(z) = Af_1(z) + Bf_2(z)$, then $\gamma(t) = A\gamma_1(t) + B\gamma_2(t)$, where $\gamma_1(t)$ and $\gamma_2(t)$ are Borel transforms for the entire functions of exponential type $f_1(z)$ and $f_2(z)$ respectively.

Example 6.1. If $f(z) = Ae^{az}$, then $\gamma(t) = \dfrac{A}{t-a}$.

Example 6.2. If

$$f(z) = \sin z = \frac{1}{2i}\Big(e^{iz} - e^{-iz}\Big), \quad \text{then} \quad \gamma(t) = \frac{1}{2i}\frac{1}{t-i} - \frac{1}{2i}\frac{1}{t+i} = \frac{1}{t^2+1}.$$

Example 6.3. If

$$f(z) = \cos z = \frac{1}{2}\Big(e^{iz} + e^{-iz}\Big), \quad \text{then} \quad \gamma(t) = \frac{1}{2}\Big(\frac{1}{t-i} + \frac{1}{t+i}\Big) = \frac{t}{t^2+1}.$$

Example 6.4. If $f(z) = 1 + e^z$, then $\gamma(t) = \dfrac{1}{t} + \dfrac{1}{t-1}$.

There are connections between the growth of $|f(z)|$ along the rays $\arg z = \varphi$ and the convex hull of the set of singularities of $\gamma(t)$.

6.2. Convex sets. The supporting function. A subset D of the complex plane is called *convex*, if for any $z, \tilde{z} \in D$ we have $\alpha z + (1-\alpha)\tilde{z} \in D$ for all $\alpha \in [0,1]$.

Let D be a closed convex set, and let ω_φ denote the ray $\arg z = \varphi$. Let $z = x+iy \in D$. The orthogonal projection of the point z to the ray ω_φ is the point $x\cos\varphi + y\sin\varphi = \operatorname{Re}(ze^{-i\varphi})$. The function

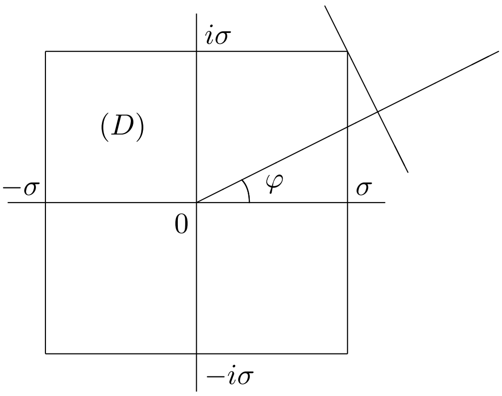
$$K(\varphi) = \max_{z\in D} \operatorname{Re}(ze^{-i\varphi}), \quad 0 \le \varphi \le 2\pi,$$

is called the *supporting function* of the convex set D. Clearly, for each point $z \in D$: $x\cos\varphi + y\sin\varphi \le K(\varphi)$, and for at least one point $z \in D$ we have the equality here. Hence, for any φ the set D lies on one side of the line $x\cos\varphi + y\sin\varphi = K(\varphi)$, and D has at least one common point with this line. This line is called a *supporting line.*

Example 6.5. Let $D = \{z : |z| \le \sigma\}$. Then $K(\varphi) \equiv \sigma$.

Example 6.6. If $D = [-i\sigma, i\sigma]$ is a segment of the imaginary axis, then $K(\varphi) = \sigma|\sin\varphi|$.

Example 6.7. Let $D = \{z : |\operatorname{Re} z| \le \sigma, |\operatorname{Im} z| \le \sigma\}$. Then $K(\varphi) = \sigma\sqrt{2}\cos(\pi/4 - |\varphi|)$, $|\varphi| \le \pi/4$. By symmetry of D, the function $K(\varphi)$ is periodic with the period $\pi/2$ (see fig.).



Theorem 6.1. *1) The supporting function $K(\varphi)$ of a bounded closed convex set D is continuous.*

2) Let $K_1(\varphi)$ and $K_2(\varphi)$ be the supporting functions of D_1 and D_2 respectively. If $K_1(\varphi) = K_2(\varphi)$ for all φ, then $D_1 = D_2$.

Proof. 1) Take φ_1 and φ_2. Let $z_1 \in D$ be such that

$$K(\varphi_1) = \max_{z\in D} \operatorname{Re}(ze^{-i\varphi_1}) = \operatorname{Re}(z_1e^{-i\varphi_1}).$$

Then

$$K(\varphi_1) - K(\varphi_2) = \operatorname{Re}(z_1e^{-i\varphi_1}) - \max_{z\in D} \operatorname{Re}(ze^{-i\varphi_2})$$

$$\le \operatorname{Re}(z_1e^{-i\varphi_1}) - \operatorname{Re}(z_1e^{-i\varphi_2}) = \operatorname{Re}(z_1(e^{-i\varphi_1} - e^{-i\varphi_2})) \le |z_1||e^{-i\varphi_1} - e^{-i\varphi_2}|.$$

Denote $B = \max\limits_{z\in D} |z|$, then $K(\varphi_1) - K(\varphi_2) \le B|e^{-i\varphi_1} - e^{-i\varphi_2}|$. Analogously we obtain $K(\varphi_2) - K(\varphi_1) \le B|e^{-i\varphi_2} - e^{-i\varphi_1}|$, and consequently,

$$|K(\varphi_1) - K(\varphi_2)| \le B|e^{-i\varphi_1} - e^{-i\varphi_2}|.$$

Thus, $|K(\varphi_1) - K(\varphi_2)| < \varepsilon$, if $|\varphi_1 - \varphi_2| < \delta = \delta(\varepsilon)$, and we have proved that the supporting function $K(\varphi)$ is continuous.

2) Suppose that $D_1 \neq D_2$. Let for definiteness there exists a point $z_0 \in D_2$ such that $z_0 \notin D_1$. Let $z_1 \in D_1$ be such that

$$\min_{z \in D_1} |z_0 - z| = |z_0 - z_1|.$$

Assume that the segment $\overline{z_1 z_0}$ is inclined to the real axis with an angle φ_0. It is easy to check that $K_1(\varphi_0) \neq K_2(\varphi_0)$, and we arrive at the second assertion of the theorem. $\square$

6.3. The integral representation for an entire function of exponential type. Let M be a bounded subset of the complex plane. The intersection D of all closed convex sets, which contain M, is called the *convex hull* of the set M. The set D is the smallest closed convex set that contains M.

Let $f(z)$ be an entire function of exponential type and let $\gamma(t)$ be its Borel transform. The convex hull $\bar{D}$ of the set of the singularities of the function $\gamma(t)$ is called the *conjugate diagram* of the function $f(z)$. As was mentioned above, if $f(z)$ is of degree κ, then $\gamma(t)$ is analytic for $|t| > \kappa$, and on the circle $|t| = \kappa$ the function $\gamma(t)$ has at least one singular point. Hence, the conjugate diagram $\bar{D}$ of $f(z)$ lies in the circle $|z| \leq \kappa$ and has at least one common point with its boundary $|t| = \kappa$.

Let $\bar{D}$ be the conjugate diagram of $f(z)$, and $K(\varphi)$ be the supporting function of the set $\bar{D}$. Obviously,

1) $\max\limits_{0 \leq \varphi \leq 2\pi} K(\varphi) = \kappa$, and if $z_0 = \kappa e^{i\varphi_0} \in \bar{D}$, then $\max\limits_{0 \leq \varphi \leq 2\pi} K(\varphi) = K(\varphi_0)$.

2) $K(\varphi) \geq -\kappa$.

Example 6.8. For the function $f(z) = Ae^{az}$ the Borel transform has the form $\gamma(t) = \dfrac{A}{t-a}$, and consequently, $\bar{D} = \{t : t = a\}$ contains only the point $t = a$.

Example 6.9. For the function $f(z) = \sin z$ one has $\gamma(t) = \dfrac{1}{t^2+1}$, and $\bar{D} = [-i, i]$ is a segment of the imaginary axis.

Example 6.10. Let $f(z) = \cosh z + \sin z$. Then $\gamma(t) = \dfrac{t}{t^2-1} + \dfrac{1}{t^2+1}$, hence $\bar{D}$ is the square with vertexes in the points: $1, -1, i, -i$.

The following theorem gives an integral representation for entire functions of exponential type.

Theorem 6.2. *Let $f(z)$ be an entire function of exponential type, let $\gamma(t)$ be the Borel transform for $f(z)$, and let $\bar{D}$ be the conjugate diagram of $f(z)$. Then*

$$f(z) = \frac{1}{2\pi i} \int_C \gamma(t) e^{zt}\, dt, \tag{6.4}$$

where C is a simply closed contour such that $\bar{D} \subset \operatorname{int} C$.

Proof. We have

$$J := \frac{1}{2\pi i} \int_C \gamma(t) e^{zt}\, dt = \frac{1}{2\pi i} \int_{|t|=R} \gamma(t) e^{zt}\, dt, \quad R > \kappa.$$

For $|t| = R$, the Laurent series (6.3) converges uniformly, and consequently,

$$J = \sum_{n=0}^{\infty} b_n \frac{1}{2\pi i} \int_{|t|=R} \frac{e^{zt}}{t^{n+1}}\, dt = \sum_{k=0}^{\infty} \frac{b_n}{n!} z^n = f(z).$$

Theorem 6.2 is proved. □

Corollary 6.1. *Let* $K(\varphi)$ *be the supporting function of the conjugate diagram* $\bar{D}$ *for the function* $f(z)$*. Then for each* $\varepsilon > 0$,

$$|f(re^{i\varphi})| < A(\varepsilon)e^{(K(-\varphi)+\varepsilon)r}, \tag{6.5}$$

where $A(\varepsilon)$ *does not depend on* r *and* φ*.*

Proof. Let $\bar{D}_\varepsilon = \{z : |z-t| \le \varepsilon, t \in \bar{D}\}$ be the ε - extension of $\bar{D}$. Then the function $K(\varphi) + \varepsilon$ is the supporting function of the set $\bar{D}_\varepsilon$. Consider the contour $C_\varepsilon := \partial \bar{D}_\varepsilon$ with counterclockwise circuit. According to Theorem 6.2,

$$f(z) = \frac{1}{2\pi i} \int_{C_\varepsilon} \gamma(t)e^{zt}\,dt,$$

and consequently,

$$|f(re^{i\varphi})| \le \frac{l}{2\pi} \max_{t\in C_\varepsilon} |\gamma(t)| \exp\Big(r \max_{t\in C_\varepsilon} \operatorname{Re}(te^{i\varphi})\Big),$$

where l is the length of C_ε. Since $\max\limits_{t\in C_\varepsilon} \operatorname{Re}(te^{i\varphi}) = K(-\varphi) + \varepsilon$, we get (6.5). □

6.4. The indicator function. Polya's theorem. For an entire function $f(z)$ of exponential type we introduce the following characteristic:

$$h(\varphi) = \overline{\lim_{r\to\infty}} \frac{\ln|f(re^{i\varphi})|}{r}, \quad 0 \le \varphi \le 2\pi.$$

The function $h(\varphi)$ is called the *indicator function* or the *indicator* of $f(z)$. The indicator describes the growth of the function along a ray $\arg z = \varphi$.

Example 6.11. Let $f(z) = e^{(a-ib)z}$. Then $f(re^{i\varphi}) = \exp((a\cos\varphi + b\sin\varphi)r)$, and consequently, the indicator has the form $h(\varphi) = a\cos\varphi + b\sin\varphi$. Such an indicator is called trigonometric.

Theorem 6.3. *Let* $f(z)$ *be an entire function of exponential type with the indicator* $h(\varphi)$, *and let* $\gamma(t)$ *be the Borel transform of* $f(z)$*. In the half-plane*

$$\operatorname{Re}(te^{i\varphi_0}) > h(\varphi_0) \tag{6.6}$$

the function $\gamma(t)$ *is analytic, and*

$$\gamma(t) = \int_0^{\infty \exp(i\varphi_0)} f(z)e^{-zt}\,dz \tag{6.7}$$

(*the integration in (6.7) is performed along the ray* $\arg z = \varphi_0$).

Proof. Consider the half-plane

$$\operatorname{Re}(te^{i\varphi_0}) > h(\varphi_0) + \delta, \quad \delta > 0. \tag{6.8}$$

Along the ray $\arg z = \varphi_0$ we have the estimate $|f(z)| < A(\varepsilon)e^{(h(\varphi_0)+\varepsilon)r}$, $\varepsilon > 0$, and consequently,

$$|f(z)e^{-zt}| < A(\varepsilon)\exp((h(\varphi_0)+\varepsilon)r - r\operatorname{Re}(te^{i\varphi_0})).$$

Therefore, if t belongs to the half-plane (6.8), then $|f(z)e^{-zt}| < A(\varepsilon)\exp(-(\delta-\varepsilon)r)$. For $0<\varepsilon<\delta$ we obtain that the integral

$$\int_0^{\infty\,\exp(i\varphi_0)} f(z)e^{-zt}\,dt \tag{6.9}$$

converges in the half-plane (6.8) absolutely and uniformly. Hence, the integral (6.9) represents an analytic function in the half-plane (6.8). Since $\delta>0$ is arbitrary, this integral represents an analytic function in the half-plane (6.6). It remains to show that the integral (6.9) is equal to $\gamma(t)$.

For a fixed $\varepsilon>0$ we choose $R_0(\varepsilon)>0$ such that

$$|f(z)| < B(\varepsilon)e^{(\kappa+\varepsilon)|z|}, \quad |z|>R_0(\varepsilon), \tag{6.10}$$

where κ is the degree of $f(z)$. Let $R>R_0(\varepsilon)$, and let t lie in the half-plane

$$\mathrm{Re}(te^{i\varphi_0}) > \kappa+\varepsilon+h, \quad h>0. \tag{6.11}$$

We have

$$\int_0^{\infty\,\exp(i\varphi_0)} f(z)e^{-zt}\,dz = A_R + B_R,$$

where

$$A_R = \int_0^{R\,\exp(i\varphi_0)} f(z)e^{-zt}\,dz, \quad B_R = \int_{R\,\exp(i\varphi_0)}^{\infty\,\exp(i\varphi_0)} f(z)e^{-zt}\,dz.$$

By virtue of (6.10), (6.11), we get

$$|B_R| < B(\varepsilon)\int_R^\infty e^{-hr}\,dr \to 0 \quad \text{for} \quad R\to\infty.$$

On the segment $[0, Re^{i\varphi_0}]$, the series (6.1) converges uniformly. Then

$$A_R = \sum_{n=0}^\infty \frac{b_n}{n!}\int_0^{R\exp(i\varphi_0)} z^n e^{-zt}\,dz.$$

Since

$$\int_0^{\infty\,\exp(i\varphi_0)} z^n e^{-zt}\,dz = \frac{n!}{t^{n+1}},$$

in the half-plane (6.11) we have

$$|\gamma(t)-A_R| = \Big|\sum_{n=0}^\infty \frac{b_n}{n!}\int_{R\exp(i\varphi_0)}^{\infty\exp(i\varphi_0)} z^n e^{-zt}\,dz\Big| \le \sum_{n=0}^\infty \frac{|b_n|}{(\kappa+\varepsilon)^{n+1}}\,\frac{(\kappa+\varepsilon)^{n+1}}{n!}\int_R^\infty r^n e^{-(\kappa+\varepsilon+h)r}\,dr.$$

Thus, it is sufficient to show that

$$a_{n,R} := \frac{(\kappa+\varepsilon)^{n+1}}{n!}\int_R^\infty r^n e^{-(\kappa+\varepsilon+h)r}\,dr \to 0, \quad R\to\infty,$$

uniformly in n. It is easy to calculate:

$$\int_R^\infty r^n e^{-(\kappa+\varepsilon+h)r}\,dr = e^{-(\kappa+\varepsilon+h)R}\frac{n!}{(\kappa+\varepsilon+h)^{n+1}}\sum_{j=0}^n \frac{((\kappa+\varepsilon+h)R)^j}{j!}.$$

Hence,

$$a_{n,R} = e^{-(\kappa+\varepsilon+h)R}\Big(\frac{\kappa+\varepsilon}{\kappa+\varepsilon+h}\Big)^{n+1}\sum_{j=0}^{n}\frac{((\kappa+\varepsilon+h)R)^j}{j!} < e^{-hR} \to 0, \quad R\to\infty.$$

Thus, (6.7) is valid in the half-plane (6.11). Since the integral (6.9) represents an analytic function in the larger half-plane (6.6), formula (6.7) holds also in the half-plane (6.6). Theorem 6.3 is proved. □

Corollary 6.2. *The following relation holds*

$$K(-\varphi) \le h(\varphi). \tag{6.12}$$

Proof. Indeed, the function $\gamma(t)$ is analytic outside $\bar{D}$, and consequently, $\gamma(t)$ is analytic in the half-plane $\mathrm{Re}(te^{i\varphi_0}) > K(-\varphi_0)$. The boundary of this half-plane is the supporting line for $\bar{D}$, which is orthogonal to the ray $\arg t = -\varphi_0$. The function $\gamma(t)$ has singularities on the supporting line. On the other hand, the function $\gamma(t)$ is analytic in the half-plane (6.6). Hence, $K(-\varphi_0) \le h(\varphi_0)$, and Corollary 6.2 is proved. □

Theorem 6.4 (Polya). *Let $f(z)$ be the entire function of exponential type with the indicator $h(\varphi)$, let $\bar{D}$ be its conjugate diagram and let $K(\varphi)$ be the supporting function of $\bar{D}$. Then*

$$h(\varphi) = K(-\varphi). \tag{6.13}$$

Proof. It follows from (6.5) that $h(\varphi) \le K(-\varphi)$. Along with (6.12) this yields (6.13). □

The following theorem follows from the properties of supporting function $K(\varphi)$.

Theorem 6.5. *The indicator function $h(\varphi)$ is continuous, 2π-periodic, and*

$$\max_{0\le\varphi\le 2\pi} h(\varphi) = \kappa.$$

Moreover, $h(\varphi) \ge -\kappa$.

According to Polya's theorem and Theorem 6.1 the indicator $h(\varphi)$ is the supporting function for the set $D := \{z : \bar{z} \in \bar{D}\}$, where $\bar{D}$ is the conjugate diagram of the corresponding entire function $f(z)$. Indeed,

$$h(\varphi) = K(-\varphi) = \max_{z\in\bar{D}} \mathrm{Re}(ze^{i\varphi}) = \max_{z\in D} \mathrm{Re}(\bar{z}e^{i\varphi}) = \max_{z\in D} \mathrm{Re}(ze^{-i\varphi}).$$

This set D is called the *indicator diagram* of the function $f(z)$. The shape of the indicator diagram indicates the growth of the entire function in different directions.

Example 6.12. For the function $f(z) = \sin z$ we have: $\rho = 1, \sigma = 1$, $\bar{D} = [-i, i]$ is a segment of the imaginary axis, $K(\varphi) = |\sin\varphi|$, $h(\varphi) = |\sin\varphi|$.

Remark 6.1. Sometimes it is convenient to define the indicator function in another way:

$$h(\varphi) = \overline{\lim_{r\to\infty}} \frac{\ln|f(re^{i\varphi})|}{r^\rho}.$$

For more details about properties of indicator functions see [4].

6.5. The indicator function of the derivative. Let $f(z)$ be an entire function of exponential type, and let D be the indicator diagram for $f(z)$.

Theorem 6.6. *Suppose that one of the following conditions is fulfilled:*
(i) the origin is an interior point of D, *i.e.* $0 \in int D$;
(ii) the origin does not belong to D;
(iii) the origin is an interior point of a segment that is a part of the boundary of D.
Then the indicator of the derivative $f'(z)$ *is equal to the indicator of* $f(z)$.

Proof. Obviously, D can be replaced with $\bar{D}$. Let $\gamma(t)$ and $\gamma_1(t)$ are the Borel transforms of f and f', respectively. Since

$$\gamma(t) = \sum_{n=0}^{\infty} \frac{b_n}{t^{n+1}}, \quad \gamma_1(t) = \sum_{n=0}^{\infty} \frac{b_{n+1}}{t^{n+1}},$$

we have

$$\gamma_1(t) = t\gamma(t) - f(0). \tag{6.14}$$

It follows from (6.14) that the functions $\gamma_1(t)$ and $\gamma(t)$ have the same singularities except maybe the origin $t = 0$. Under the hypothesis of the theorem, we get that the conjugate diagram of $f'(z)$ coincides with the conjugate diagram of $f(z)$. Therefore, the supporting function of $f'(z)$ is equal to the supporting function of $f(z)$, and by Theorem 6.4, the indicator of $f'(z)$ is equal to the indicator of $f(z)$. □

Remark 6.2. If the origin is a “corner” of D, then the indicators of $f(z)$ and $f'(z)$ may happen to be different. For example, let $f(z) = 1 + e^z$. Then $f'(z) = e^z$,

$$\gamma(t) = \frac{1}{t} + \frac{1}{t-1}, \quad \gamma_1(t) = \frac{1}{t-1}.$$

Both the indicator and conjugate diagrams of $f(z)$ coincide with the segment $[0, 1]$, while for $f'(z)$ they are the point 1.

Theorem 6.7. *Let* $\gamma_m(t)$ *be the Borel transform for* $f^{(m)}(z)$, $m \geq 0$. *Then*

$$\gamma_m(t) = t^m\gamma(t) - t^{m-1}f(0) - t^{m-2}f'(0) - \ldots - f^{(m-1)}(0). \tag{6.15}$$

Indeed, using (6.14), one can easily derive (6.15) by induction. □

6.6. Introduction to the operational calculus. In the sequel, $f(z) =: \gamma(t)$ means that $\gamma(t)$ is the Borel transform of $f(z)$. We consider the linear differential equation with constant coefficients

$$a_0y^{(n)}(x) + a_1y^{(n-1)}(x) + \ldots + a_ny(x) = \varphi(x) \tag{6.16}$$

under the initial conditions

$$y(0) = b_0,\ y'(0) = b_1,\ \ldots,\ y^{(n-1)}(0) = b_{n-1}. \tag{6.17}$$

Let $\varphi(z)$ be an entire function of exponential type, and $\varphi(z) =: \gamma^*(t)$. We shall seek a solution $y(x)$ of (6.16), (6.17) in the class of entire functions of exponential type. Put

$y(z) =: \gamma(t)$. By Theorem 6.7 we have $y^{(m)}(z) =: \gamma_m(t) = t^m\gamma(t) - t^{m-1}b_0 - \ldots - b_{m-1}$. Taking in (6.16) the Borel transforms, one gets

$$a_0(t^n\gamma(t)-t^{n-1}b_0-\ldots-b_{n-1})+a_1(t^{n-1}\gamma(t)-t^{n-2}b_1-\ldots-b_{n-2})+\ldots+a_n\gamma(t) = \gamma^*(t). \quad (6.18)$$

Consequently,

$$\gamma(t) = \frac{\gamma^*(t) + P_{n-1}(t)}{P(t)}, \quad P(t) := a_0t^n + \ldots + a_n, \tag{6.19}$$

where $P_{n-1}(t)$ is a known polynomial of degree $\le n-1$. It follows from (6.19) that $\gamma(t)$ is analytic at infinity and $\gamma(\infty) = 0$. Hence, $\gamma(t) = \sum_{k=0}^{\infty} \frac{c_k}{t^{k+1}}$. Consequently, the function $y(x) = \sum_{k=0}^{\infty} \frac{c_k}{k!}x^k$ is the solution of the Cauchy problem (6.16), (6.17).

The described method of solving the Cauchy problem for differential equations is the central point of the operational calculus. This method allows one to reduce the solution of a differential equation to the solution of an algebraic (so-called *operational*) equation (6.18). One can find $y(x)$ from $\gamma(t)$ also by the formula

$$y(x) = \frac{1}{2\pi i}\int_C \gamma(t)e^{xt}\,dt$$

with the help of the residue calculus.

Example 6.12. Consider the Cauchy problem:

$$y''(x) - 2y'(x) + y(x) = 0, \;\; y(0) = 0, \;\; y'(0) = 1.$$

We get $y(x) =: \gamma(t)$, $y'(x) =: t\gamma(t)$, $y''(x) =: t^2\gamma(t) - 1$, and the operational equation has the form $t^2\gamma(t) - 1 - 2t\gamma(t) + \gamma(t) = 0$. This yields

$$\gamma(t) = \frac{1}{(t-1)^2}, \quad y(x) = \frac{1}{2\pi i}\int_C \frac{e^{xt}}{(t-1)^2}\,dt = xe^x.$$

Example 6.13. Let us find the solution of the following problem:

$$y'' + y' + y = \cos x, \quad y(0) = 0, \; y'(0) = 1.$$

We get $y =: \gamma(t)$, $y' =: t\gamma(t)$, $y''(x) =: t^2\gamma(t) - 1$, $\cos x =: t/(t^2+1)$. The operational equation has the form $t^2\gamma(t) - 1 + t\gamma(t) + \gamma(t) = t/(t^2+1)$. This yields $\gamma(t) = \frac{1}{t^2+1}$, $y(x) = \sin x$.

7. Estimates of the modulus of an entire function from below

Estimates of the modulus of an entire function from below play an important role in the theory of entire functions and its applications, for example, in the spectral theory.

First we consider entire functions of order $0 \le \rho < 1/2$. According to Corollary 3.3, such functions can be represented (up to a multiplicative constant) in the canonical form:

$$f(z) = z^m \prod_{k=1}^{\infty}\Big(1 - \frac{z}{z_k}\Big), \tag{7.1}$$

where the numbers $z_k \neq 0$ are zeros of $f(z)$. Then

$$\min_{|z|=r} |f(z)| \geq r^m \prod_{k=1}^{\infty} \left(1 - \frac{r}{|z_k|}\right) = \varphi(r), \tag{7.2}$$

where

$$\varphi(z) = z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{|z_k|}\right). \tag{7.3}$$

We note that the entire function $\varphi(z)$ has the same order $\rho < 1/2$, as $f(z)$, since the zeros of these functions have the same convergence exponent τ. By virtue of Corollary 5.1, the function $\varphi(z)$ is not bounded on the half-line $z > 0$. Therefore, there exists a sequence $r_n \to \infty$ such that $|\varphi(r_n)| \to \infty$. Consequently, $\min_{|z|=r_n} |f(z)|$ tends to infinity as $n \to \infty$. Thus, we have proved the following assertion.

Theorem 7.1. *For each entire function $f(z)$ of order $0 \leq \rho < 1/2$, there exists a sequence of circles $|z| = r_n$, $r_n \to \infty$, such that $\min\limits_{|z|=r_n} |f(z)| \to \infty$ as $n \to \infty$.*

This assertion is not valid for entire functions of order $\rho \geq 1/2$. For example, the entire function $f(z) = \cos\sqrt{z}$ has the order $\rho = 1/2$, and it is bounded for $z > 0$.

Theorem 7.2. *Let $f(z)$ be an entire function of order $0 < \rho < 1/2$. Then for each $\varepsilon > 0$, there exists a sequence $r_n \to \infty$, for which*

$$\min_{|z|=r_n} |f(z)| > \exp(r_n^{\rho-\varepsilon}).$$

Proof. Consider the function $F(z) = \varphi(z)\exp(-\sigma(-z)^{\rho-\varepsilon})$, $\varepsilon > 0$, $0 < \sigma < 1$, where $\varphi(z)$ is defined by (7.3). The function $F(z)$ is analytic in the whole complex plane without the half-line $z \geq 0$. We choose $(-z)^{\rho-\varepsilon}$ such that $(-z)^{\rho-\varepsilon}$ is real for $z < 0$. Since $M_\varphi(r) = \varphi(-r)$, there exists a sequence $r_n \to \infty$ such that $\varphi(-r_n) > \exp(r_n^{\rho-\varepsilon})$. Then

$$F(-r_n) > \exp(r_n^{\rho-\varepsilon} - \sigma r_n^{\rho-\varepsilon}) = \exp((1-\sigma) r_n^{\rho-\varepsilon}). \tag{7.4}$$

Let us show that $|F(z)|$ is not bounded on the half-line $z > 0$. Indeed, suppose on the contrary, that $|F(z)| < C$ for $z > 0$. Using similar arguments as in the proof of the Phragmen-Lindelöf theorem, one can prove (using $\rho < 1/2$) that $|F(z)| < C$ for all z. But this contradicts to (7.4). Thus, there exists a sequence $r_n \to \infty$ such that $|F(r_n)| > 1$. Then

$$|\varphi(r_n)| > |\exp(\sigma(-r_n)^{\rho-\varepsilon})| = \exp(\sigma r_n^{\rho-\varepsilon} \cos\pi(\rho-\varepsilon)),$$

and consequently, $|\varphi(r_n)| > \exp(r_n^{\rho-2\varepsilon})$, for sufficiently large n. □

We now consider the function $\sin z$ which is an entire function of exponential type with the indicator $h(\varphi) = |\sin\varphi|$.

Theorem 7.3. *1) For all $z = re^{i\varphi}$ one has $|\sin z| \leq \exp(|\mathrm{Im}\, z|) = \exp(h(\varphi)r)$;*
2) Denote $G_\varepsilon = \{z : |z - k\pi| \geq \varepsilon, k = 0, \pm1, \pm2, \ldots\}$. Then for $z \in G_\varepsilon$,

$$|\sin z| \geq A(\varepsilon)\exp(h(\varphi)r), \quad A(\varepsilon) > 0.$$

Proof. 1) Since $\sin z = (e^{iz} - e^{-iz})(2i)^{-1}$, we get

$$|\sin z| \le \frac{e^{\mathrm{Im}\, z} + e^{-\mathrm{Im}\, z}}{2} \le \exp(|\mathrm{Im}\, z|) = \exp(h(\varphi)r).$$

2) Since the function $|\sin z|$ is even and π-periodic, it is sufficient to consider $z \in \Pi \cup \Pi_\varepsilon$, where

$$\Pi = \Big\{z : -\frac{\pi}{2} \le \mathrm{Re}\, z \le \frac{\pi}{2},\ \mathrm{Im}\, z > 1\Big\},\ \ \Pi_\varepsilon = \Big\{z : -\frac{\pi}{2} \le \mathrm{Re}\, z \le \frac{\pi}{2},\ 0 \le \mathrm{Im}\, z \le 1,\ |z| \ge \varepsilon\Big\}.$$

Put $F(z) := |\sin z| \exp(-\mathrm{Im}\, z)$. If $z \in \Pi$, we get

$$F(z) = \frac{|e^{2iz} - 1|}{2} \ge \frac{1 - e^{-2\mathrm{Im}\, z}}{2} > \frac{1 - e^{-2}}{2}.$$

Further, for $z \in \Pi_\varepsilon$ we have $F(z) \ge \min\limits_{\xi \in \Pi_\varepsilon} F(\xi) > 0$. Thus, taking

$$A(\varepsilon) = \min\Big\{\frac{1 - e^{-2}}{2},\ \min_{\xi \in \Pi_\varepsilon} F(\xi)\Big\}$$

we arrive at the second assertion of the theorem. $\square$

The next assertion we provide without proof (see [5] for details).

Theorem 7.4. *Let* $f(z) = \prod\limits_{n=1}^{\infty}\Big(1 - \dfrac{z^2}{z_n^2}\Big)$, $z_n > 0$, $\lim\limits_{n\to\infty} n/z_n = \sigma < \infty$. *Then* $f(z)$ *is an entire function of exponential type with the indicator* $h(\varphi) = \pi\sigma|\sin\varphi|$. *Moreover, there exist* $r_n \to \infty$ *such that*

$$|f(r_n e^{i\varphi})| > \exp((h(\varphi) - \varepsilon)r_n), \quad k > n_0(\varepsilon), \quad \varepsilon > 0.$$

8. Meromorphic functions

A function $f(z)$ is called *meromorphic*, if it is analytic in the whole complex plane with exception of its poles. For example, if $f_1(z) \not\equiv 0$ and $f_2(z)$ are entire, then $f(z) = f_2(z)/f_1(z)$ is a meromorphic function. In particular, each rational fraction is a meromorphic function. If a meromorphic function has a countable infinite set of poles $\{z_k\}_{k\ge1}$, then $\lim\limits_{k\to\infty} |z_k| = \infty$, and we can enumerate them as follows: $|z_k| \le |z_{k+1}|$ (each pole is counted according to its multiplicity).

We mention that each meromorphic function $f(z)$ has the form $f_2(z)/f_1(z)$, where $f_1(z)$ and $f_2(z)$ are entire functions. Indeed, by Theorem 3.1, one can construct an entire function $f_1(z)$ such that its zeros coincide with $\{z_k\}$. Then the function $f_2(z) = f(z)f_1(z)$ is entire and $f(z) = f_2(z)/f_1(z)$.

We note that one can also define the order, type, etc. of a meromorphic function and extend most of the results presented in this note to the meromorphic case – for details we refer the reader to the textbook [9].

The gamma-function. Consider the function

$$\Gamma(z) = e^{-Cz} \frac{1}{z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}}, \tag{8.1}$$

where

$$C = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0,5772...$$

is the Euler constant. The function $\Gamma(z)$ is called the *gamma-function*. Clearly, $\Gamma(z)$ is a meromorphic function with simple poles in the points $z = -k$ $(k = 0, 1, 2, \ldots)$. The function $\Gamma(z)$ has no zeros. We provide some properties of the gamma-function.

Let us show that Γ satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1. \tag{8.2}$$

For this purpose we consider the functions

$$f_n(z) = e^{-Cz} \frac{1}{z \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right) e^{-z/k}} = \frac{n! \exp\left\{ -Cz + \sum_{k=1}^{n} \frac{z}{k} \right\}}{z(z+1)\cdots(z+n)}. \tag{8.3}$$

Clearly, $\lim_{n\to\infty} f_n(z) = \Gamma(z)$. We calculate

$$\frac{z f_n(z)}{f_n(z+1)} = (z+n+1) \exp\left(C - \sum_{k=1}^{n} \frac{1}{k} \right) = \frac{z+n+1}{n} \exp\left(C - \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \right).$$

For $n \to \infty$ we obtain

$$\frac{z\Gamma(z)}{\Gamma(z+1)} = 1,$$

i.e. $\Gamma(z+1) = z\Gamma(z)$. It follows from (8.3) for $z = 1$ that

$$f_n(1) = \frac{1}{n+1} \exp\left(-C + \sum_{k=1}^{n} \frac{1}{k} \right) = \frac{n}{n+1} \exp\left(-C + \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \right),$$

and consequently, $\Gamma(1) = \lim_{n\to\infty} f_n(1) = 1$.

Using (8.3) we get

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z \exp\left(-Cz + \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) z \right)}{z(z+1)\ldots(z+n)},$$

hence,

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\ldots(z+n)}. \tag{8.4}$$

Formula (8.4) is called the *Gauss-Euler formula* for the gamma-function.

Let us show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{8.5}$$

Indeed,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{k=1}^{\infty} \Big(1 - \frac{z^2}{k^2}\Big). \tag{8.6}$$

Since

$$z \prod_{k=1}^{\infty} \Big(1 - \frac{z^2}{k^2}\Big) = \frac{1}{\pi}\Big(\pi z \prod_{k=1}^{\infty} \Big(1 - \frac{z^2\pi^2}{\pi^2 k^2}\Big)\Big) = \frac{1}{\pi} \sin \pi z$$

(see formula (3.31)) and since $-z\Gamma(-z) = \Gamma(1-z)$, from (8.6) we derive (8.5).

Let us show that for $\Gamma(z)$ the following integral representation is valid

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt, \quad \operatorname{Re} z > 0, \tag{8.7}$$

where $t^{z-1} = \exp((z-1)\ln t)$. Indeed, $|t^{z-1}e^{-t}| = e^{-t}t^{x-1}$, $z = x+iy$. Hence, the integral in (8.7) converges absolutely for each z in the half-plane $\operatorname{Re} z > 0$. The integral $\int_0^{\infty} t^{z-1}e^{-t}\, dt$ converges uniformly on each bounded closed subset of this half-plane. Since the function $t^{z-1}e^{-t}$ is entire with respect to z for each $t > 0$, the function $\int_0^{\infty} e^{-t}t^{z-1}\, dt$ is analytic for $\operatorname{Re} z > 0$.

We note that $\Big(1 - \dfrac{t}{n}\Big)^n \to e^{-t}$ for $n \to \infty$ uniformly on each finite segment, and $0 < \Big(1 - \dfrac{t}{n}\Big)^n < e^{-t}$ for $0 < t < n$. Therefore,

$$\lim_{n\to\infty} \int_0^n \Big(1 - \frac{t}{n}\Big)^n t^{z-1}\, dt = \int_0^{\infty} t^{z-1}e^{-t}\, dt \quad \text{for} \quad \operatorname{Re} z > 0.$$

On the other hand, since

$$J_n = \int_0^n \Big(1 - \frac{t}{n}\Big)^n t^{z-1}\, dt = n^z \int_0^1 \tau^n (1-\tau)^{z-1}\, d\tau,$$

integration by parts yields

$$J_n = \frac{n!n^z}{z(z+1)\cdots(z+n-1)} \int_0^1 \tau^{z+n-1} d\tau = \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

For $n \to \infty$, by virtue of (8.4), we arrive at (8.7).

For the gamma-function the following Stirling formula holds:

$$\Gamma(z) \sim \sqrt{2\pi} z^{-1/2} e^{z\ln z - z}, \quad \text{for} \quad z \to \infty,\ |\arg z| < \pi - \delta,\ \delta > 0. \tag{8.8}$$

(see, for example, [6]). In particular, for $z = n+1$ we obtain from (8.8):

$$\Gamma(n+1) = n! \sim \sqrt{2\pi}(n+1)^{n+\frac{1}{2}} e^{-n-1}.$$

Since

$$(n+1)^{n+\frac{1}{2}} = n^{n+\frac{1}{2}} \Big(1 + \frac{1}{n}\Big)^n \Big(1 + \frac{1}{n}\Big)^{\frac{1}{2}} \sim e n^{n+\frac{1}{2}},$$

we get the following asymptotical formula for $n!$:

$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n, \quad n \to \infty.$$

Let us show that $\frac{1}{\Gamma(z)}$ is an entire function of order $\rho = 1$ and infinite type. Indeed,

$$\frac{1}{\Gamma(z)} = ze^{Cz}\prod_{k=1}^{\infty}\Big(1+\frac{z}{k}\Big)e^{-z/k}.$$

By virtue of Lemma 3.1 and Theorem 3.10, $\frac{1}{\Gamma(z)}$ is an entire function, and its order is $\rho = 1$, since the convergence exponent of the sequence of its zeros is $\tau = 1$, and $g(z) = Cz$ is a polynomial of the first degree. The function $\frac{1}{\Gamma(z)}$ is of maximal type $(\sigma = \infty)$, i.e. there do not exist constants $A > 0$ and $B > 0$ such that for all z,

$$\Big|\frac{1}{\Gamma(z)}\Big| < Ae^{B|z|}. \tag{8.9}$$

Indeed, by virtue of (8.8), $\frac{1}{\Gamma(x)} \sim \sqrt{2\pi}x^{-\frac{1}{2}}e^{x\ln x - x}$, $x > 0$, $x \to \infty$, which shows that (8.9) cannot be fulfilled for all z.

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