ON THE EQUATION div $u = g$ AND BOGOVSKIĬ'S OPERATOR IN SOBOLEV SPACES OF NEGATIVE ORDER

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Dedicated to Philippe Clément on the Occasion of his 60-th Birthday

Abstract. Consider the divergence problem with homogeneous Dirichlet data on a Lipschitz domain. Two approaches for its solutions in the scale of Sobolev spaces are presented. The first one is based on Calderón-Zygmund theory, whereas the second one relies on the Stokes equation with inhomogeneous data.

1. Introduction

The solution of many problems in hydrodynamics requires a thorough understanding of the structure of the solutions of the equation $\operatorname{div} u = g$ for a given scalar valued function g. Hence, given a domain $\Omega \subset \mathbb{R}^n$, quite a few authors (see e.g. [Cat61], [Lad69], [Neč67], [SŠ73], [Bog79], [Bog80], [Pil83], [Sol83], [vW90], [BS90], [FS94], [Gal94], [Soh01]) dealt with the problem

$$\begin{cases} \operatorname{div} u = g \text{ in } \Omega \\ u_{|\partial\Omega} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

There are several approaches to prove the existence of a solution to problem (1.1), see [Bog79], [Gal94], [vW90] and [Pil83]. Observe also that the solution to this problem is not unique.

Bogovskiĭ proved the existence and a-priori estimates for a solution to (1.1) in the scale of Sobolev spaces of positive order provided $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, $n \geq 2$ and $g \in L^p(\Omega)$ satisfies $\int_\Omega g = 0$. Here $1 < p < \infty$. His approach is based on an explicit representation formula for u on star shaped domains. This representation of u as a singular integral allows to apply Calderón-Zygmund theory and estimates for u in Sobolev spaces of positive order follow thus by this theory.

In this paper we prove that Bogovskiĭ's solution operator B can be extended continuously to an operator acting from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)^n$ provided $s > -2 + 1/p$. Our approach is based on properties of the adjoint kernel of K, K associated to B, see also [BS90], [MM05]. Results of this type are quite useful in the study of the Navier-Stokes flow past rotating obstacles, see e.g [GHH04]. Note that the case $s = -1$ was already considered by Borchers and Sohr in [BS90]; see however [FS94] and the footnote on page 180 of Galdi's book [Gal94].

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A completely different approach to equation (1.1) is based on estimates for the solution of the inhomogeneous Stokes system

$$\begin{array}{rcll} -\Delta u + \nabla p &=& f & \text{in } \Omega \\ \operatorname{div} u &=& g & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

see Section 3. Setting $f = 0$ and $Bg := u$, where (u, p) is the solution to problem (1.2), one obtains by this approach in particular estimates for the solution to problem (1.1) provided $g \in \widehat{W}^{1,p}(\Omega)$ or $g \in \widehat{W}^{-1,p}(\Omega)$. In Section 3 we extend this approach to $g \in \widehat{W}^{s,p}(\Omega)$ for all $s \in [-1, 1]$. For related problems as e.g. groundwater flow we refer to [CL93].

2. Approach by an explicit representation formula

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $1 < p < \infty$. For $s \geq 0$ we denote by $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ the usual Sobolev spaces, see e.g. [Tri95]. Furthermore, let $W_0^{s,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}$. For $s < 0$ we set

$$W^{s,p}(\Omega) := (W_0^{-s,p'}(\Omega))' \text{ and } W_0^{s,p}(\Omega) := (W^{-s,p'}(\Omega))',$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Note that $C_c^\infty(\Omega)$ is dense in $W_0^{s,p}(\Omega)$ for all $s \in \mathbb{R}$.

Our first proposition relies on the fact that for bounded and star shaped domains with respect to a ball K a solution to problem (1.1) can be written as a singular integral. More precisely, choose $\omega \in C_c^\infty(K)$ with $\int_K \omega = 1$ and define for $g \in C_c^\infty(\Omega)$

$$(B_i g)(x) := \int_\Omega g(y) \frac{x_i - y_i}{|x - y|^n} \int_0^\infty \omega\Big(x + r\frac{x - y}{|x - y|}\Big)(|x - y| + r)^{n-1} \, \mathrm{d}r \, \mathrm{d}y, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)$ and $i = 1, \ldots, n$.

Then the following holds.

Proposition 2.1. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded and star shaped domain with respect to some ball. Let $g \in L^p(\Omega)$. Then $B := (B_1, \ldots, B_n)$ satisfies*

$$BC_c^\infty(\Omega) \subset C_c^\infty(\Omega)^n$$

and

$$\nabla \cdot Bg = g - \omega \int_\Omega g \text{ for } g \in L^p(\Omega).$$

Moreover, for $s > -2 + \frac{1}{p}$, B can be continuously extended to a bounded operator from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)^n$.

Proof. The case $s \geq 0$ was already treated by Bogovskiĭ in [Bog79]. There it is proved that $B_i C_c^\infty(\Omega) \subset C_c^\infty(\Omega)^n$. Moreover, by Calderón-Zygmund theory for the singular integral (2.1) one obtains the assertion for $s \geq 0$. For a detailed proof see [Gal94, Lemma III.3.1].

In the following we prove the assertion for $s < 0$ by duality. In fact, the kernel of the adjoint B_i^* of the operator B_i is given by

$$\begin{aligned}
K_i^*(x, x-y) &= -\frac{x_i - y_i}{|x-y|^n} \int_0^\infty \omega(y - r\frac{x-y}{|x-y|^n})(|x-y| + r)^{n-1}\, \mathrm{d}r \\
&= -(x_i - y_i) \int_1^\infty \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \\
&= -(x_i - y_i) \int_0^\infty \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r + (x_i - y_i) \int_0^1 \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \\
&= -\frac{x_i - y_i}{|x-y|^n} \int_0^\infty \omega(x - r\frac{x-y}{|x-y|}) r^{n-1}\, \mathrm{d}r + (x_i - y_i) \int_0^1 \omega(x - r(x-y)) r^{n-1}\, \mathrm{d}r \\
&=: K_{i,\mathrm{sing}}^*(x, x-y) + K_{i,\mathrm{bdd}}^*(x, x-y).
\end{aligned}$$

Since

$$|\partial_{x_j} K_{i,\mathrm{bdd}}^*(x, x-y)| \le C, \qquad i, j = 1, \ldots, n, x \in \Omega, y \in \mathbb{R}^n,$$

it follows that the operator associated to the kernel $K_{i,\mathrm{bdd}}^*$ continuously maps $L^p(\Omega)$ into $W^{1,p}(\Omega)$. We thus consider in the following the contribution of $K_{i,\mathrm{sing}}^*$. Similarly as in the proof of [Gal94, Lemma III.3.1], the kernel $\partial_{x_j} K_{i,\mathrm{sing}}^*$ can be decomposed in a weakly singular kernel $K_{i,\mathrm{sing,w}}^*$ and a Calderón-Zygmund kernel $K_{i,\mathrm{sing,CZ}}^*$. More precisely, $\partial_{x_j} K_{i,\mathrm{sing}}^*$ can be rewritten as

$$\partial_{x_j} K_{i,\mathrm{sing}}^* = K_{i,\mathrm{sing,w}}^* + K_{i,\mathrm{sing,CZ}}^*,$$

with

$$\begin{aligned}
K_{i,\mathrm{sing,w}}^*(x, x-y) &= -\frac{x_i - y_i}{|x-y|^n} \int_0^\infty (\partial_{x_j}\omega)(x - r\frac{x-y}{|x-y|}) r^{n-1}\, \mathrm{d}r, \\
K_{i,\mathrm{sing,CZ}}^*(x, x-y) &= \frac{-\delta_{ij}}{|x-y|^n} \int_0^\infty \omega(x - r\frac{x-y}{|x-y|}) r^{n-1}\, \mathrm{d}r \\
&\quad + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty (\partial_{x_j}\omega)(x - r\frac{x-y}{|x-y|}) r^n\, \mathrm{d}r.
\end{aligned}$$

Note that $K_{i,\mathrm{sing,CZ}}^*$ satisfies the following properties

(a) $K_{i,\mathrm{sing,CZ}}^*(x, z) = \alpha^{-n} K_{i,\mathrm{sing,CZ}}^*(x, \alpha z)$, $x \in \Omega$, $z \in \mathbb{R}^n$, $\alpha > 0$,

(b) $\int_{|z|=1} K_{i,\mathrm{sing,CZ}}^*(x, z)\, \mathrm{d}z = 0$, $x \in \Omega$,

(c) $|K_{i,\mathrm{sing,CZ}}^*(x, z)| \le C$, $x \in \Omega$, $|z| = 1$.

It follows from classical Calderón-Zygmund theory [CZ56], [Ste93] that

$$B_i^* \in \mathcal{L}(L^p(\Omega), W^{1,p}(\Omega)).$$

Moreover,

$$B_i^* \in \mathcal{L}(W_0^{\tilde{s},p}(\Omega), W^{\tilde{s}+1,p}(\Omega)), \quad \tilde{s} > 0$$

by [Gal94, Remark III.3.1] and real interpolation. Since $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ for $-1 + \frac{1}{p} < s < \frac{1}{p}$, we obtain

$$B \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s+1,p}(\Omega)^n), \qquad -2 + \frac{1}{p} < s < -1.$$

The remaining cases finally follow by real interpolation. □

Remark 2.2. (a) *It should be emphasized that in the above proposition, the operator B is defined for all $g \in L^p(\Omega)$ whereas Bogovskiĭ* [Bog79]*, von Wahl* [vW90] *and Galdi* [Gal94] *constructed solutions to the problem (1.1). The latter is only possible if $\int_\Omega g = 0$. Hence, B may be regarded as extension of the solution operator to problem (1.1). However, if $\int_\Omega g \neq 0$, then Bg is not a solution to (1.1).*

(b) *The idea of using the adjoint kernel to prove estimates for B in Sobolev spaces of negative order is quite natural and was already used in* [BS90] *for the case $s = -1$. More recently, this approach was also reconsidered by* [MM05].

(c) *There is a considerable difference between B_i and its adjoint B_i^*. As $B_i C_c^\infty(\Omega) \subset C_c^\infty(\Omega)$, this does not hold true for its adjoint.*

(d) *The above proof shows that $B \in \mathcal{L}(W_0^{s,p}(\Omega), W^{s+1,p}(\Omega)^n)$ for $s \leq -1$.*

Bounded and locally Lipschitz domains have the remarkable property that they can be written as a finite union of star shaped domains. This gives us the possibility to carry over mapping properties of the operator B, originally defined on star shaped domains, to locally Lipschitz domains. For convenience and to fix notation we first state a result concerning the decomposition of Lipschitz domains into star shaped domains. For a proof of this result we refer to [Gal94, Lemma III.3.4].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and locally Lipschitz domain. Then there exist $m \in \mathbb{N}$ and an open cover $\mathcal{G} = \{G_i : i \in \{1, \dots, m\}\}$ of $\overline{\Omega}$ such that for $1 \leq i \leq m$ the set $\Omega_i = \Omega \cap G_i$ is star shaped with respect to some ball and $\Omega = \bigcup_{i=1}^m \Omega_i$.*

Moreover, there exist $\phi_i \in C_c^\infty(G_i)$, $m_i \in \mathbb{N}$, $\theta_{i,k} \in C_c^\infty(\Omega_i)$ and $\psi_{i,k} \in C_c^\infty(\overline{\Omega})$ ($i \in \{1, \dots, m\}$, $k \in \{1, \dots, m_i\}$) such that

$$P_i g := \phi_i g + \sum_{k=1}^{m_i} \theta_{i,k} \int_\Omega \psi_{i,k} g, \quad g \in C_c^\infty(\Omega)$$

satisfies $P_i g \in C_c^\infty(\Omega_i)$ and $\int_\Omega P_i g = 0$. In addition, if $\int_\Omega g = 0$ we get a decomposition of g by $g = \sum_{i=1}^m P_i g$.

In order to define a solution operator to (1.1) for bounded, locally Lipschitz domains we reconsider the operators P_i.

Lemma 2.4. *Let $1 < p < \infty, s \in \mathbb{R}$ and let Ω_i and P_i be defined as in Lemma 2.3 for $i = 1, \dots, m$. Then*

$$P_i \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s,p}(\Omega_i)),\ s \in \mathbb{R},\ 1 < p < \infty.$$

Proof. Let $i \in \{1, \dots, m\}$. Consider first the case where $s \geq 0$. Then there exists $C > 0$ such that

$$\|P_i g\|_{W_0^{s,p}(\Omega_i)} \leq C\|g\|_{W_0^{s,p}(\Omega)}, \quad g \in W_0^{s,p}(\Omega).$$

In order to prove the remaining cases, let $s > 0$, $g \in C_c^\infty(\Omega)$ and $v \in W^{s,p'}(\Omega_i)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\begin{aligned}
|\langle P_i g, v\rangle| &= |\int_{\Omega_i} \phi(x)g(x)v(x) + \sum_{k=1}^m \theta_{i,k}(x)v(x)\int_\Omega \psi_{i,k}(y)g(y)\,\mathrm{d}y\,\mathrm{d}x| \\
&= |\int_{\Omega_i} g(x)\phi(x)v(x)\,\mathrm{d}x + \sum_{k=1}^m \int_\Omega \psi_{i,k}(y)g(y)\int_\Omega \theta_{i,k}(x)v(x)\,\mathrm{d}x\,\mathrm{d}y| \\
&\leq \|g\|_{W_0^{-s,p}(\Omega)}\|\phi v\|_{W^{s,p'}(\Omega)} + \|g\|_{W_0^{-s,p}(\Omega)}\sum_{k=1}^m \|\psi_{i,k}\int_\Omega \theta_{i,k}(x)v(x)\,\mathrm{d}x\|_{W^{s,p'}(\Omega_i)} \\
&\leq C\|g\|_{W_0^{-s,p}(\Omega)}\|v\|_{W^{s,p'}(\Omega_i)}
\end{aligned}$$

where C is some constant independent of g and v. □

The following theorem is the main result of this paper. Besides its interest in its own, there are many applications of the use of Bogovskiĭ's operator in Sobolev spaces of negative order; see e.g. the recent paper [GHH04].

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a locally Lipschitz boundary. Then there exists $B : C_c^\infty(\Omega) \to C_c^\infty(\Omega)^n$ such that*

$$\nabla \cdot Bg = g, \quad g \in L^p(\Omega) \text{ with } \int_\Omega g = 0.$$

Moreover, B can be extended continuously to a bounded operator from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)^n$ provided $s > -2 + \frac{1}{p}$.

Proof. Let $g \in C_c^\infty(\Omega)$. Consider the decomposition of Ω and the associated operators P_i defined as in Lemma 2.3. Then

$$\sum_{i=1}^m P_i g = g \text{ provided } \int_\Omega g(x)\,\mathrm{d}x = 0. \tag{2.2}$$

Denote the operator defined in Proposition 2.1 acting on Ω_i by B_i and set $Bg := \sum_{i=1}^m B_i P_i g$. Then by Proposition 2.1 and Lemma 2.4, $B \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s+1,p}(\Omega)^n)$ for all $s \geq 0$ since $B_i P_i C_c^\infty(\Omega) \subset C_c^\infty(\Omega_i)^n$.

Again, by Proposition 2.1 and Lemma 2.4

$$\begin{aligned}
|\langle Bg, v\rangle| &= |\int_\Omega \sum_{i=1}^m B_i P_i g(x)v(x)\,\mathrm{d}x| = |\sum_{i=1}^m \int_{\Omega_i} B_i P_i g(x)v(x)\,\mathrm{d}x| \\
&\leq C\sum_{i=1}^m \|P_i g\|_{W_0^{-s,p}(\Omega_i)}\|v\|_{W^{s,p'}(\Omega_i)} \\
&\leq C\|g\|_{W_0^{-s,p}(\Omega)}\|v\|_{W^{s,p'}(\Omega)}, \quad g \in C_c^\infty(\Omega),\ v \in W^{s,p'}(\Omega).
\end{aligned}$$

Finally, by (2.2) we obtain

$$\nabla \cdot Bg = \sum_{i=1}^{m} \nabla \cdot B_i P_i g = \sum_{i=1}^{m} P_i g = g, \quad g \in L^p(\Omega), \ \int_\Omega g(x)\,\mathrm{d}x = 0.$$

□

3. Approach by the inhomogeneous Stokes equation

We start this section by considering the problem

$$(3.1) \qquad \begin{aligned} -\Delta u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= g && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ is a bounded domain with boundary $\partial\Omega \in C^2$. Let $1 < p, p' < \infty$ such that $1 = \frac{1}{p} + \frac{1}{p'}$.

We then set $L_0^p(\Omega) := \{f \in L^p(\Omega) : \int_\Omega f = 0\}$ and for $s \in [0,1]$ let $\widehat{W}^{s,p}(\Omega) := W^{s,p}(\Omega) \cap L_0^p(\Omega)$ equipped with the norm in $W^{s,p}(\Omega)$. Furthermore, we define $\widehat{W}^{-s,p}(\Omega) := (\widehat{W}^{s,p'}(\Omega))'$ equipped with the usual dual norm.

Note that for $g \in \widehat{W}^{1,p}(\Omega)$ we have

$$\|g\|_{\widehat{W}^{-1,p}(\Omega)} = \sup_{v \in \widehat{W}^{1,p}(\Omega) \setminus \{0\}} \frac{|\langle g, v\rangle|}{\|v\|_{W^{1,p'}(\Omega)}} \leq \|g\|_{W_0^{-1,p}(\Omega)}.$$

The following proposition is due to Farwig and Sohr [FS94].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $1 < p < \infty$. Then there exists a bounded operator $R : \widehat{W}^{1,p}(\Omega) \to W^{2,p}(\Omega)^n \cap W_0^{1,p}(\Omega)^n$ such that* $\operatorname{div} Rg = g$. *Moreover, R satisfies the following estimates:*

(a) $\|Rg\|_{L^p(\Omega)} \leq C\|g\|_{\widehat{W}^{-1,p}(\Omega)} \leq C\|g\|_{W_0^{-1,p}(\Omega)}$,

(b) $\|Rg\|_{W^{2,p}(\Omega)} \leq C\|g\|_{W^{1,p}(\Omega)}$.

Here $C > 0$ is a constant depending on Ω and p only.

Setting $f = 0$ and $Rg := u$, where (u,p) is the solution of (3.1) the assertion above is a direct consequence of the unique solvability of the problem (3.1) with $f \in (L^p(\Omega))^n$ and $g \in \widehat{W}^{1,p}(\Omega)$.

Similarly, we obtain the fact that $R \in \mathcal{L}(L_0^p(\Omega), (W^{1,p}(\Omega))^n)$ from the unique solvability of the problem (3.1) for $f \in W^{-1,p}(\Omega)^n$ and $g \in L_0^p(\Omega)$. In fact, since (u,p) is the unique solution of (3.1), the operator R given in Proposition 3.1 may be thus extended from $\widehat{W}^{1,p}(\Omega)$ to $L_0^p(\Omega)$. Unique solvability of (3.1) in the given setting was first proved by Cattabriga [Cat61] for the case $n = 3$ and by Galdi and Simader [GS90] for general $n \geq 2$. See also [KS91] for a different proof. We summarize these facts in the next proposition.

Proposition 3.2. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Then for every $f \in W^{-1,p}(\Omega)^n$ and $g \in L_0^p(\Omega)$ there exists a unique solution $(u,p) \in W_0^{1,p}(\Omega)^n \times L_0^p(\Omega)$ of (3.1) satisfying the inequality*

$$\|\nabla u\|_{L^p(\Omega)} + \|p\|_{L^p(\Omega)} \leq C(\|f\|_{W^{-1,p}(\Omega)^n} + \|g\|_{L^p(\Omega)})$$

for some constant $C = C(\Omega, n, p)$.

Noting that $(L_0^{p'}(\Omega))' = L_0^p(\Omega)$, the following lemma implies that $L_0^p(\Omega)$ is dense in $\widehat{W}^{-1,p}(\Omega)$. The proof is standard and therefore omitted.

Lemma 3.3. *Let X, Y be Banach spaces. Assume that X is densely embedded in Y and that X is reflexive. Then the closure of Y' is X'.*

Combining the above results, we may define a solution operator for the divergence problem (1.1) in the following spaces

$$R: \begin{cases} \widehat{W}^{1,p}(\Omega) & \to \quad W^{2,p}(\Omega)^n \cap W_0^{1,p}(\Omega)^n \\ L_0^p(\Omega) & \to \quad W_0^{1,p}(\Omega)^n \\ \widehat{W}^{-1,p}(\Omega) & \to \quad L^p(\Omega)^n \end{cases}$$

The following result gives additional mapping properties of R in the scale of Sobolev spaces.

Theorem 3.4. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $s \in [-1, 1]$. Then there exists a bounded linear operator $R : \widehat{W}^{s,p}(\Omega) \to W_0^{s+1}(\Omega)^n$ such that $\operatorname{div} Rg = g$ for all $g \in \widehat{W}^{s,p}(\Omega)$.*

Proof. The cases $s = 1$ and $s = -1$ follow from Proposition 3.1. Consider next the case where $0 \le s < 1$. Denote by K the set of all constant functions over Ω. Then we may identify the spaces $L_0^p(\Omega)$ with $L^p(\Omega)/K$ and $\widehat{W}^{1,p}(\Omega)$ with $W^{1,p}(\Omega)/K$, respectively. As K is a one-dimensional vector space, it follows from [Tri95, Section 1.17.2, Remark 1] that

$$(L_0^p(\Omega), \widehat{W}^{1,p}(\Omega))_{s,p} = (L^p(\Omega)/K, W^{1,p}(\Omega)/K)_{s,p} = (L^p(\Omega), W^{1,p}(\Omega))_{s,p}/K = \widehat{W}^{s,p}(\Omega).$$

This implies the assertion provided $0 \le s < 1$.

In order to prove the remaining case where $-1 < s < 0$, note that $(L_0^{p'}(\Omega), \widehat{W}^{1,p'}(\Omega))'_{s,p'} = (L_0^p(\Omega), \widehat{W}^{-1,p}(\Omega))_{s,p}$; see e.g. [Tri95, Section 1.11.2]. Hence,

$$\widehat{W}^{-s,p}(\Omega) = (L_0^{p'}(\Omega), \widehat{W}^{1,p'}(\Omega))'_{s,p'} = (L_0^p(\Omega), \widehat{W}^{-1,p}(\Omega))_{s,p}$$

and the proof is complete. □

Remark 3.5. *The assertions of Theorems 2.5 and 3.4 remain valid also for the complex interpolation spaces. In fact, the above mapping properties of B and R in the scale of Sobolev spaces hold true also in the scale of the spaces $H^{s,p}(\Omega)$ and $\widehat{H}^{s,p}(\Omega)$, respectively.*

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