J. Math. Kyoto Univ. (JMKYAZ)
34-3 (1994) 539-584

Compactness in Boltzmann's equation via Fourier integral operators and applications. III

dedicated to the memory of Ron DiPerna

By

P. L. Lions

I. Introduction

This work is the continuation of Parts I [55] and II [56]. In particular, a general introduction to Boltzmann's equation (and kinetic models) can be found in [55] together with a description of the main goals of this series. References can be found in the bibliography here which is a combined bibliography of Parts I-III. We also keep the same notations as in [55], [56].

Let us begin by recalling the general form of the Boltzmann's equation where we include force terms

$$\frac{\partial f}{\partial t}+v\cdot \nabla_x f+F\cdot \nabla_v f=Q(f,f) \ , \qquad x\in\mathbf{R},\ v\in\mathbf{R}^N,\ t\geq 0 \tag{1}$$

where $N\geq 2, f$ -the unknown- is always assumed to be a nonnegative (scalar) function on $\mathbf{R}^{2N}_{x,v}\times[0,\infty)$.

In the classical Boltzmann's equation, one takes $F\equiv 0$ and the so-called collision term (or operator) $Q(f,f)$ introduced by L. Boltzmann [11] and J. C. Maxwell [61], [62] is given by the following bilinear operator

$$Q(f,f)=Q^+(f,f)-Q^-(f,f) \ , \tag{2}$$

$$Q^+(f,f)=\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega B(v-v_*,\omega)f'f'_* \ , \tag{3}$$

$$\begin{cases} Q^-(f,f)=\displaystyle\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega B(v-v_*,\omega)ff_*=fL(f) \ , \\ L(f)=A\underset{v}{*}f \ , \end{cases} \tag{4}$$

and $A(z)=\int_{S^{N-1}}B(z,\omega)d\omega\,(z\in\mathbf{R}^N)$, $f_*=f(x,v_*,t)$, $f'=f(x,v',t)$, $f'_*=f(x,v',$

Communicated by Prof. T. Nishida, July 15, 1993

$t)$, $v'=v-(v-v_*,\ \omega)\,\omega$, $v'_*=v_*+(v-v_*,\ \omega)\,\omega$, $v'_*=v_*+(v-v_*,\ \omega)\,\omega$. Let us recall that here, as in [55], [56], we denote indifferently by $a\cdot b$ or $(a,\ b)$ the usual scalar product of $a,\ b\in\mathbf{R}^N$.

The so-called collision kernel B that enters the operator Q is a given function on $\mathbf{R}^N\times S^{N-1}$. We shall always assume (at least) that B satisfies

$$B\in L^1(B_R\times S^{N-1}) \quad \text{for all} \quad R\in(0,\ \infty)\ ,\quad B\geq 0 \tag{5}$$

where $B_R=\{z\in\mathbf{R}^N,\ |z|<R\}$, and

$$B(z,\ \omega) \quad \text{depends only on} \quad |z| \quad \text{and} \quad |(z,\ \omega)|\ , \tag{6}$$

$$\begin{cases} (1+|z|^2)^{-1}\left(\int_{z+B_R} A(v)\,dv\right)\to 0 \\ \qquad\qquad \text{as} \quad |z|\to\infty\ , \quad \text{for all} \quad R\in(0,\ \infty) \end{cases} \tag{7}$$

and we shall not recall these assumptions in all that follows.

A classical example is given by the so-called hard-spheres model where we have

$$B(z,\ \omega)=|(z,\ \omega)|\ .$$

To complete the description of (1), we now have to explain the meaning and detail the form of F. It is physically natural to add a force term to the classical Boltzmann's equation (see for instance C. Cercignani [13], [14]) and mathematically this modifies very little the analysis at least when F is a given external force i.e. a given function on $\mathbf{R}_x^N$ (or on $\mathbf{R}_x^N\times[0,\ \infty)$, or even $\mathbf{R}_{x,v}^{2N}\times[0,\ \infty))$. However, if the particles whose dynamics are described in a statistical fashion by (1), interact with a two-body force, we are naturally led to a Vlasov-like force (or self-consistent force, or mean field...) F given by

$$\begin{cases} F=-\nabla V\ , \quad V=V_0\underset{x}{*}\rho\ , \\ \rho(x,\ t)=\int_{\mathbf{R}^N} f(x,\ v,\ t)\,dv \qquad \text{on} \quad \mathbf{R}_x^N\times[0,\ \infty) \end{cases} \tag{9}$$

where V_0 is the interaction potential between the particles always assumed to be at least in $W_{loc}^{1,1}(\mathbf{R}^N)$. Of course, we might add to this force a given (external force) and we may consider as well more complicated systems with several species of particles... But these extensions do not affect the results we prove here and this is why we prefer to skip them. A case of particular physical interest corresponds to the so-called Vlasov-Poisson model where $N=3$, $V_0=\dfrac{1}{4\pi|x|}$ so that (9) becomes

$$-\Delta_x V=\rho \quad \text{on } \mathbf{R}_x^N\times[0,\ \infty)\ ,\quad \rho=\int_{\mathbf{R}^N} f\,dv \quad \text{on } \mathbf{R}_x^N\times[0,\ \infty)\ . \tag{10}$$

We shall call the composite model (1) and (9) the Vlasov-Boltzmann system (VB in short) and the above example $\left(N=3,\ V_0=\dfrac{1}{4\pi|x|}\right)$ the

Vlasov-Poisson-Boltzmann system (VPB in short). Let us finally mention an extension of the VPB system namely the Vlasov-Maxwell-Boltzmann system (VMB in short) which looks like the VPB system is a very natural model in Physics for charged particles (in plasmas, lasers...). In that case, (9) is replaced by the Lorentz force determined by the electro-magnetic field created by the particles themselves. More precisely the VMB system consists of (1) and choosing $N=3$

$$F=E(x,t)+\frac{1}{c}v\times B(x,t) \tag{11}$$

$$\frac{\partial E}{\partial t}-c\operatorname{curl} B=-j\ ,\quad \operatorname{div} B=0 \qquad \text{on} \quad \mathbf{R}^3\times(0,\infty) \tag{12}$$

$$\frac{\partial B}{\partial t}+c\operatorname{curl} E=0\ ,\quad \operatorname{div} E=\rho \qquad \text{on} \quad \mathbf{R}^3\times(0,\infty) \tag{13}$$

$$\begin{cases} \rho=\int_{\mathbf{R}^3} f\,dv\ ,\quad j_k=\int_{\mathbf{R}^3} fv_k dv \\ \qquad\qquad (1\leq k\leq 3) \qquad \text{on} \quad \mathbf{R}^3\times(0,\infty)\ , \end{cases} \tag{14}$$

where c denotes the speed of light.

Of course, all these systems have to be complemented with initial conditions. In the case of the VB system, one simply prescribes f at time $t=0$ i.e.

$$f|_{t=0}=f_0 \qquad \text{on} \quad \mathbf{R}^{2N}_{x,v} \tag{15}$$

where $f_0\geq 0$ satisfies some bounds detailed below. And in the case of the VMB system, we add to (15) initial conditions for E, B i.e.

$$E|_{t=0}=E_0\ ,\quad B|_{t=0}=B_0 \qquad \text{on} \quad \mathbf{R}^3_x \tag{16}$$

with the usualy compatibility condition

$$\operatorname{div} E_0=\rho_0=\int_{\mathbf{R}^3} f_0\,dv \qquad \text{on} \quad \mathbf{R}^3_x\ . \tag{17}$$

We state in section II below our main existence and compactness results concerning the VB system. We assume that f_0 satisfies

$$\begin{cases} \iint_{\mathbf{R}^{2N}} f_0(1+\omega(x)+|v|^2+|\log f_0|)\,dxdv \\ \qquad +\iint_{\mathbf{R}^{2n}} \rho_0(x)\,|V_0|(x-y)\,\rho_0(y)\,dx\,dy<\infty\ , \end{cases} \tag{18}$$

where ω satisfies

$$\omega\geq 0\ ,\quad (1+\omega)^{1/2} \ \text{ is Lipschitz on } \ \mathbf{R}^N\ ,\quad e^{-\omega}\in L^1(\mathbf{R}^N) \tag{19}$$

(typical examples are $\omega(x)=|x|^2$, $|x|$, $(1+|x|^2)^{\alpha/2}$ with $0<\alpha\leq 2$...). We also make some regularity assumptions on V_0 that are detailed and discussed in section II: let us only mention at this stage that these conditions hold in the case of the VPB system i.e. $N=3$, $V_0=\frac{1}{4\pi|x|}$.

We then present in section II a result that states the existence of global weak solutions - whose precise definitions are given in section II, let us only mention that they correspond to the formulation introduced in Part II [56] for Boltzmann's equation and which is a refinement of the notion of renormalized solutions introduced in R. J. DiPerna and P. L. Lions [25], [26]. This global existence result is thus the analogue of the existence results shown in [25], [26] and refined in Parts I and II [55], [56].

The proof of this global existence result is given in section III and of course as in [25], [26], [30] relies upon compactness and stability results that are also presented in section II and proved in section III. These results concern sequences of weak (possibly approximate) solutions of the VB system denoted by f^n, corresponding to initial conditions $f_0^n\geq 0$ which satisfy the following natural uniform bounds

$$\text{(20)}\qquad \begin{cases} \sup_{n\geq 1}\Big\{\int\int_{\mathbf{R}^{2N}} f_0^n(1+|v|^2+\omega(x)+|\log f_0^n|)\,dx\,dv \\ \qquad +\int\int_{\mathbf{R}^{2N}}\rho_0^n(x)\,|V_0(x-y)|\,\rho_0^n(y)\,dx\,dy\Big\}<\infty\ , \end{cases}$$

where, of course, $\rho_0^n=\int_{\mathbf{R}^N} f_0^n dv$. As we shall see in section II, under simple conditions on V_0, these bounds imply similar bounds uniform in t on the solutions f^n namely

$$\text{(21)}\qquad \begin{cases} \sup_{n\geq 1,\,t\in[0,T]}\Big\{\int\int_{\mathbf{R}^{2N}} f^n(1+|v|^2+\omega(x)+|\log f^n|)\,dx\,dv \\ \qquad +\int\int_{\mathbf{R}^{2N}}\rho^n(x)\,|V_0(x-y)|\,\rho^n(y)\,dx\,dy\Big\}<\infty \end{cases}$$

for all $T\in(0,\infty)$. And without loss of generality, extracting subsequences if necessary, we may assume that f_0^n, f^n converge weakly in L^1 (respectively weakly in $L^1(\mathbf{R}^{2N})$, $L^1(\mathbf{R}^{2N}\times(0,T))$ for all $T\in(0,\infty)$), to some f_0, f respectively. Our first result states that f is also a global weak solution of the VB system satisfying (15) and is thus the analogue of the result shown in [25], [26] on Boltzmann's equation. But we want to point out that the extra Vlasov term requires a new argument which in fact can be seen as a simplification of the original proof made in [25] for Boltzmann's equation. The proof uses heavily renormalization techniques (and in particular the results of R. J. DiPerna and the author [29] on ordinary differential equations and linear first-order equations with nonsmooth coefficients): indeed, one weakly passes to the limit in the renormalized equations, then renormalize the resulting limit equation and finally let the first renormalization go to the identity. Except for this new idea, the main compactness argument is, as in [25], the compactness of velocity averages (or macroscopic quantities) in L^1.

The second compactness result shows that, if f_0^n converges strongly in L^1 to f^n, then f^n converges to f strongly in $L^1(\mathbf{R}^{2N}_{x,v})$ uniformly in $t\in[0,T]$ (for all $T\in(0,\infty)$). This result is thus the analogue for VB systems of the one shown in Part II [56] for Boltzmann's equation. As in [56] the proof relies upon the a.e. compactness of the gain term (of the collision operator) and some renormalization techniques for linear first-order equations.

At this stage, it is worth mentioning that all the "compactness-stability" and existence results are shown under conditions on V_0 which include variants of the following (crucial) one

$$(22)\qquad \begin{cases} D_x^2(V_0*\rho)\in L^1_{loc}(\mathbf{R}^N) \qquad \text{if} \quad \rho\geq 0\ , \\ \int_{\mathbf{R}^N}\rho(|\log\rho|+1+\omega(x))dx<\infty\ . \end{cases}$$

This condition is satisfied, for instance, in the case of the VP system where $N=3$, $V_0=\frac{1}{|x|}$: indeed, in that case, (22) holds in view of classical results on Riesz transforms - see, for exemple, E. Stein [68]. It is also worth noting that the condition (22) is essentially the condition needed in the proof of P. L. Lions and B. Perthame [57] for Vlasov systems (without the collision terms) on the propagation of high moments in v and the regularity of solutions.

Next, in section IV, we consider the coupled VMB system (1), (11)-(14) and we prove the existence of "very weak solutions" for general initial conditions (15)-(16)) (f_0, E_0, B_0) satisfying (17), $f_0\geq 0$ a.e. and

$$(23)\qquad \iint_{\mathbf{R}^6} f_0(1+\omega(x)+|v|^2+|\log f_0|)\,dx\,dv+\int_{\mathbf{R}^3}|E_0|^2+|B_0|^2dx<\infty\ .$$

In fact, we introduce for this purpose a new notion of weak solution that is a bit weaker than the notion introduced in [25] (namely, the notion of renormalized solution).

Finally, in section V, we study the so-called Boltzmann-Dirac model (BD in short) consisting of (1) with $F=0$ and a collision term Q given by

$$(24)\qquad \begin{cases} Q=\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega B(v-v_*,\omega) \\ \qquad\cdot\{f'f'_*(1-\varepsilon f)(1-\varepsilon f_*)+ff_*(1-\varepsilon f')(1-\varepsilon f')\} \end{cases}$$

where B satisfies the same conditions than in the above Boltzmann models and $\varepsilon>0$ is a (small) physical parameter. The physical background on such a model can be found, for instance, in S. Chapman and T. G. Cowling [17]: let us simply indicate that this phenomenological model aims to incorporate quantum effects such as the Pauli exclusion principle in the statistical description of possible collisions. From a mathemanical viewpoint, this collision operator presents some advantages over the classical Boltzmann's operator since one expects, at least formally, solutions f to satisfy

$$(25) \qquad 0 \leq f \leq 1/\varepsilon \quad \text{on} \quad \mathbf{R}^{2N}_{x,v} \times [0, \infty]$$

at least if the initial condition (15) f_0 satisfies the same constraints. In other words, L^∞ bounds are available. A general study of this BD equation was performed in J. M. Dolbeault [34] that yields the existence, uniqueness and further properties (conservations, $\varepsilon \to 0$...) of solutions at least when B satisfies in addition

$$(26) \qquad \iint_{\mathbf{R}^N \times S^{N-1}} dz\, d\omega\, B(z, \omega) = \int_{\mathbf{R}^N} A(z)\, dz < \infty ,$$

a condition that excludes the hard-spheres model. We show here a general existence result of bounded solutions which relies upon a weak compactness result for solutions of (BD) corresponding to initial conditions which weakly converge in $L^1(\mathbf{R}^{2N}_{x,v})$ (and satisfy uniform natural bounds described in section V). In turn, this compactness result shown in section V is deduced from "compactification properties" of various nonlinear terms that appear in Q. Namely, we show that for a bounded sequence of solutions f^n then, under appropriate conditions on B, the following quantities are compact in $L^1_{loc}(\mathbf{R}^{2N}_{x,v} \times (0, \infty))$

$$(27) \qquad \begin{cases} \displaystyle\int_{\mathbf{R}^N}\int_{S^{N-1}} dv_* dw\, B(v - v_*, \omega) F^n \\ \qquad \text{where} \quad F^n = f^{n\prime} f^{n\prime}_*, f^{n\prime} f^n_*, f^{n\prime}_* f^n_*, f^{n\prime} f^{n\prime}_* f^n_* . \end{cases}$$

In fact the first term is (essentially) the gain term we analysed in Part I [55] for which we proved such an "automatic" compactness. The two other quadratic terms namely $f^{n\prime} f^n_*$ and $f^{n\prime}_* f^n_*$ are simple adaptations of the preceding one. We also have to prove the weak convergence in L^1_{loc} (and strong convergence in L^1_{loc} of velocity averages) of the trilinear term (27) where $F^n = f^n_* f^{n\prime} f^{n\prime}_*$.

II. Existence and compactness results for Vlasov-Boltzmann systems

In this section we investigate the VB system (1) - (9) (when $N \geq 2$) and the collision term $Q(f, f)$ is given by (2) - (4) and B satisfies (5) - (7). We shall not recall these assumptions below.

We first collect some ("classical") a priori estimates. First of all, using the classical identity

(28) $$\begin{cases} \int_{\mathbf{R}^N} Q(f,f)\varphi(v)\,dv \\ \quad = \frac{1}{4}\int\int_{\mathbf{R}^{2N}} dv\,dv_* \int_{S^{N-1}} d\omega B(f'f'_* - ff_*)\,[\varphi+\varphi_* - \varphi' - \varphi'_*] \end{cases}$$

we deduce the following local conservations of mass, momentum and kinetic energy

(29) $$\frac{\partial}{\partial t}\rho + \mathrm{div}_x(j) = 0 \quad \text{in} \quad \mathbf{R}^N \times (0,\infty)$$

(30) $$\frac{\partial}{\partial t} j_k + \mathrm{div}_x\left(\int_{\mathbf{R}^N} v\, v_k f\,dv\right) = 0 \quad \text{in} \quad \mathbf{R}^N \times (0,\infty)$$

(31) $$\begin{cases} \frac{\partial}{\partial t}\left(\int_{\mathbf{R}^N} f|v|^2 dv\right) + \mathrm{div}_x\left(\int_{\mathbf{R}^N} v|v|^2 f dv\right) \\ +2\mathrm{div}_x\{(V_0*\rho)j\} - 2(V_0*\rho)\,\mathrm{div}_x(j) = 0 \quad \text{in } \mathbf{R}^N \times (0,\infty)\ . \end{cases}$$

Integrating (29), (30), (31) in x over $\mathbf{R}^N$ and using (29) in (31) we deduce the following global conservations of mass, momentum and total energy

(32) $$\frac{d}{dt}\int\int_{\mathbf{R}^{2N}} f\,dx\,dv = 0 \quad \text{for} \quad t \geq 0\ ,$$

(33) $$\frac{d}{dt}\int\int_{\mathbf{R}^{2N}} f\,v_k\,dx\,dv = 0 \quad \text{for} \quad t\geq 0, \text{ for } 1\leq k\leq N\ ,$$

(34) $$\begin{cases} \frac{d}{dt}\{\int\int_{\mathbf{R}^{2N}} f|v|^2 dx\,dv + \\ \qquad + \int\int_{\mathbf{R}^{2N}} \rho(x)\,V_0(x-y)\,\rho(y)\,dx\,dy\} = 0 \quad \text{for } t\geq 0\ . \end{cases}$$

Therefore, if we assume that the initial condition f_0 (see (15)) and V_0 satisfy for some $C\geq 0$

(35) $$V_0 \in L^1_{loc}(\mathbf{R}^N)\ ,\quad V_0^-(x-y) \leq C(1+\omega(x)+\omega(y)) \quad \text{a.e. } x, y\in\mathbf{R}^N$$

(36) $$\begin{cases} \int\int_{\mathbf{R}^{2N}} f_0\,\{1+|v|^2+\omega(x)+|\log f_0|\}\,dx\,dv \\ \qquad + \int\int_{\mathbf{R}^{2N}} \rho_0(x)\,|V_0(x-y)|\,\rho_0(y)\,dx\,dy < +\infty \end{cases}$$

we deduce for the nonnegativity of f and f_0

(37) $$\begin{cases} \sup_{[0,T]} \int\int_{\mathbf{R}^{2N}} f(t)\ \{1+|v|^2+\omega(x)\}\,dx\,dv \\ \qquad + \int\int_{\mathbf{R}^{2N}} \rho(x,t)\,|V_0(x-y)|\,\rho_0(y,t)\,dx\,dy \leq C(T) \end{cases}$$

for some nonnegative constant $C(T)$ that depends only on T and on the bound

(36). Indeed, the L^1 bound on $f(t)$ is obvious in view of (32) while (34) yields, for all $t\geq 0$, because of (35)

$$\iint_{\mathbf{R}^{2N}} f(t)\,|v|^2 dx\,dv+\iint_{\mathbf{R}^{2N}} \rho(x,t)\,|V_0(x,y)|\rho(y,t)\,dx\,dy$$

$$\leq C+2\iint_{\mathbf{R}^{2N}} \rho(x,t)\,V^-{}_0(x-y)\,\rho(y,t)\,dx\,dy$$

$$\leq C+C\int_{\mathbf{R}^N} \rho(x,t)\,\omega(x)\,dx=C\Big(1+\iint_{\mathbf{R}^{2N}} f(t)\,\omega(x)\,dx\,dv\Big)\ .$$

Here and below, C denotes various constants that depend only on (36). Next, we observe that we also have

$$\frac{d}{dt}\Big(\int_{\mathbf{R}^N} f(t)\,\omega(x)\,dv\Big)+\mathrm{div}_x\Big\{\int_{\mathbf{R}^N} f(t)\,v\omega dv\Big\}$$

$$=\int_{\mathbf{R}^N} f(t)\,v\cdot\nabla\omega(x)\,dv$$

$$\leq\frac{1}{2}\int_{\mathbf{R}^N} f|v|^2 dv+\frac{1}{2}\rho(t,x)\,|\nabla\omega(x)|^2$$

$$\leq\frac{1}{2}\int_{\mathbf{R}^N} f|v|^2 dv+C+C\int_{\mathbf{R}^N} f(t)\,\omega(x)\,dv$$

in view of (19). In particular, we deduce

$$\frac{d}{dt}\iint_{\mathbf{R}^{2N}} f(t)\,\omega(x)\,dx\,dv\leq C+\frac{1}{2}\iint_{\mathbf{R}^{2N}} f(t)\,[|v|^2+\omega(x)]\,dx\,dv\ .$$

We easily obtain (37) from these inequalities applying Grönwall's lemma.

The final formal bound we wish to obtain is deduced from the (formal) entropy identity. As usual (see [13]...) it is obtained multiplying (1) by $\log f$, using (28) which yields, at least formally

$$\left\{\begin{aligned}&\frac{d}{dt}\iint_{\mathbf{R}^{2N}} f\log f\,dx\,dv\\&+\frac{1}{4}\int_{\mathbf{R}^N} dx\iint_{\mathbf{R}^{2N}} dv\,dv_*\int_{S^{N-1}} d\omega B\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}=0\ .\end{aligned}\right.\tag{38}$$

Since the second term is clearly nonnegative we deduce in particular

$$\sup_{t\geq 0}\iint_{\mathbf{R}^{2N}} f(t)\log f(t)\,dx\,dv\leq\iint_{\mathbf{R}^{2N}} f_0\log f_0\,dx\,dv\ .$$

This inequality together with (36) and (37) then implies (see Part I [55] for more details)

$$\sup_{t\in[0,T]}\iint_{\mathbf{R}^{2N}} f(t)\,|\log f(t)|\,dx\,dv\leq C(T)\tag{39}$$

where $C(T)$ denotes various nonnegative constants which depend only on T and on the bound (36).

Then, if we go back to (38), we also deduce from (39)

$$\int_0^T dt \int_{\mathbf{R}^N} dx \int\int_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} d\omega B\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*} \leq C(T) \quad . \tag{40}$$

In *conclusion*, we obtain the following (formal) bounds

$$\begin{cases} \sup_{t\in[0,T]} \int\int_{\mathbf{R}^{2N}} f(t)\, \{1+|v|^2+\omega(x)+|\log f(t)|\}\, dx dv + \\ \qquad + \int\int_{\mathbf{R}^{2N}} \rho(x,t)|V_0(x-y)|\rho(y,t)\,dx\,dy \leq C(T) \\ \int_0^T dt \int_{\mathbf{R}^N} dx \int\int_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} d\omega B\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*} \leq C(T) \end{cases} \tag{41}$$

for all $T\in(0,\infty)$.

In addition, if $\omega(x)=|x|^2$, $C(T)$ is independent of T provided we replace ω by $\omega(x-vt)$ in (41).

Before we state our main existence and stability results, we need to explain the notion(s) of weak solutions we use. First, we state some regularity conditions we need on the potential V_0:

$$\begin{cases} \text{If} \quad \varphi\geq 0,\ \varphi(1+\omega+|\log\varphi|+V_0^+*\varphi) \text{ is bounded in } L^1(\mathbf{R}^N), \\ \text{then} \quad V_0*\varphi \text{ is bounded in } L^1_{loc}(\mathbf{R}^N), \\ \nabla(V_0*\varphi) \text{ is bounded in } L^\infty(\mathbf{R}^N)+L^{\frac{N+2}{3}}(\mathbf{R}^N), \\ D^2(V_0*\varphi) \text{ is bounded in } L^1_{loc}(\mathbf{R}^N). \end{cases} \tag{42}$$

The role of this condition will be clear in the proofs of the results we state below. Let us make a few remarks about it. First of all, we observe that because of (41), the integrability conditions of (42) hold for $\varphi=\rho(t)$ (uniformly in $t\in[0,T]$, for all $T\in(0,\infty)$). This is immediate except for the integrability of $\varphi|\log\varphi|$ whose proof is given in the Appendix 1. We thus deduce that $V_0*\rho\in L^\infty(0,T;L^1_{loc}(\mathbf{R}^N_x))$, $\nabla_x(V_0*\rho)\in L^\infty(0,T;L^\infty+L^{\frac{N+2}{2}}(\mathbf{R}^N_x))$, $D^2_x(V_0*\rho)\in L^\infty(0,T;L^1_{loc}(\mathbf{R}^N_x))$ for all $T\in(0,\infty)$.

We next give a few examples showing how one can check (42). Let us mention, by the way, that (42) is (when $N=3$) exactly the condition needed in P. L. Lions and B. Perthame [57] for the obtention of moments bounds (and regularity) for solutions of Vlasov systems. Next, if $V_0=\frac{1}{|x|^\alpha}$ with $\alpha\geq 0$ ($\alpha<N$, $\alpha=0$ means $V_0=\log|x|$) then we claim that (42) holds if $0\leq\alpha\leq\min\left(\frac{2N}{N+1},\ N-2\right)$. Indeed, first of all, by a simple Fourier analysis (for example), we see that $|x|^{-\beta}*\varphi\in L^2(\mathbf{R}^N)$ if $(|x|^{-\alpha}*\varphi)\varphi\in L^1(\mathbf{R}^N)$ where $\beta=\frac{N+\alpha}{2}$. Therefore, by Hölder's inequality, $|x|^{-(1+\alpha)}*\varphi$ and thus $\nabla(V_0*\varphi)\in$

$L^p(\mathbf{R}^N)$ where $p=\frac{2\beta}{1+\alpha}=\frac{N+\alpha}{1+\alpha}$. And $p\geq\frac{N+2}{3}$ if and only if $\alpha\leq\frac{2N}{N+1}$. Next, if $\alpha<N-2$, $D_x^2 V_0\in L^{N/(\alpha+2),\infty}(\mathbf{R}^N)$ (Marcinkiewicz space) and thus $D_x^2(V_0*\varphi)\in L^{N/(\alpha+2),\infty}(\mathbf{R}^N)\subset L^1_{loc}$. The case $\alpha=N-2$ is more delicate: indeed, we have for some constant c_N

$$\frac{\partial^2}{\partial x_i\partial x_j}(V_0*\varphi)=c_N R_i R_j\varphi$$

where R_i denotes the Riesz transform $\left(\frac{\partial}{\partial x_i}(-\Delta)^{-1/2}\right)$. And we conclude that $D_x^2(V_0*\varphi)\in L^1_{loc}(\mathbf{R}^N)$ by classical Harmonic Analysis results (see for instance E. M. Stein [68]) since $\varphi(1+|\log\varphi|)\in L^1(\mathbf{R}^N)$. Observe finally that the condition $\alpha\leq\min(\frac{2N}{N+1}, N-2)$ is clearly met when $\alpha=1$, $N=3$ - the case that corresponds to the Vlasov-Poisson model (in three dimension).

We may now *define solutions* of the system (1), (9) ((VB) system): we say that $f\in C([0,\infty); L^1(\mathbf{R}^{2N}_{x,v}))\geq 0$ is a *weak* solution of (1), (9) (of the (VB) system) if for all $T\in(0,\infty)$

$$\text{(A)}\qquad\begin{cases}\sup_{t\in[0,T]}\left\{\iint_{\mathbf{R}^{2N}}f(t)\ \{1+|v|^2+\omega(x)+|\log f(t)|\}\,dx\,dv+\right.\\ \qquad\left.+\iint_{\mathbf{R}^{2N}}\rho(x,t)\,|V_0(x-y)|\,\rho(y,t)\,dx\,dy\right\}<\infty\ ,\\ \int_0^T dt\int_{\mathbf{R}^N}dx\iint_{\mathbf{R}^{2N}}dv\,dv_*\int_{S^{N-1}}d\omega B\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}<\infty\end{cases}$$

and if we have for all $m\in\mathcal{M}$

$$\text{(43)}\qquad <\frac{\partial f}{\partial t}+v\cdot\nabla_x f, m>+<F\cdot\nabla_v f, m>=<Q(f,f), m>$$

where $\mathcal{M}$ is a class of test functions (or multipliers) which was introduced in Part II [56] and that we recall now: m belongs to $\mathcal{M}$ if

$$\text{(44)}\qquad m=\varphi(x,t)\,\alpha'(p(v)\,(f-g))+\psi(x,v,t)\,\beta'(f-g)$$

where $\varphi\in C_0^\infty(\mathbf{R}^N_x\times[0,\infty))$, $\psi\in C_0^\infty(\mathbf{R}^{2N}_{x,v}\times[0,\infty))$, $\beta\in C^1([0,\infty))$ such that $\beta'(t)\,(1+t)^{1/2}$ and $\beta(t)\,(1+t)^{-\alpha}$ are bounded on $[0,\infty)$ where $\alpha=1/2$ if $N=2$, $\alpha=\frac{N-1}{N+2}$ if $N>3$. In addition, p satisfies

$$\text{(45)}\qquad\begin{cases}p\in C(\mathbf{R}^N)\ ,\quad p>0\quad\text{on}\quad\mathbf{R}^N\ ,\\ \left(A*\left(\frac{1}{p}\right)\right)(1+|v|^2)^{-1}\in C_0(\mathbf{R}^N)\ ,\quad \nabla_v\left(\frac{1}{p}\right)\in L^1(\mathbf{R}^N)\ .\end{cases}$$

And α belongs to a class $\tilde{\mathcal{B}}$ defined as follows. First of all, we define a class B consisting of those functions $\alpha\in C^1(\mathbf{R};\mathbf{R})$ satisfying $\alpha(0)=0$, α is Lipschitz on $\mathbf{R}$, $\alpha'(t)=1$ for $t>0$ large and $\alpha(t)-\alpha'(t)t$ is bounded on $\mathbf{R}$. And $\tilde{\mathscr{B}}$ consists of functions $\alpha\in C(\mathbf{R};\mathbf{R})$ such that $\alpha(0)=0$, α is Lipschitz on $\mathbf{R}$, α admits right and left derivatives for all t and is differentiable everywhere except for a finite number of points, $\alpha'(t)=1$ for $t>0$ large and $\alpha(t)-\alpha'(t)t\in L^{\infty}(\mathbf{R})$. Finally, at each nondifferentiable point t_i, we define $\alpha'(t_i)$ by choosing an arbitrary value between $\alpha'(t_{i-})$ and $\alpha'(t_{i+})$ (included).

We finally have to define the class of test functions g that we allow i.e. the space $\mathscr{A}$ given by all functions g satisfying

$$g\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x,v}))\ ,\quad g|v|\in L^{\infty}(0,T;L^1(\mathbf{R}^{2N}_{x,v}))\ ,$$

$$\frac{A*g}{1+|v|^2}\in L^{\infty}(0,T;L^{\infty}(\mathbf{R}^{2N}_{x,v}))\ ,\quad \frac{\partial g}{\partial t}+v\cdot\nabla_x g\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))\ ,$$

$$\nabla_v g\in L^1(0,T;L^p_{loc}(\mathbf{R}^N_x;L^1(\mathbf{R}^N_v)))$$

for all $T\in(0,\infty)$, where $p=2$ if $N=2$, $p=\frac{N+2}{N-1}$ if $N\geq 3$.

Remark II.1. i) One might allow more general multipliers relaxing the regularity of β, φ, ψ, choosing more elaborate functions p and β that may depend on x, t, taking different functions g inside α and β...

ii) In Part II [56], we did not require that $\alpha(t)-\alpha'(t)$ is bounded on $\mathbf{R}$. This additional assumption seems to be needed here in order to cope with the force term "$F\cdot\nabla_v f$".

iii) Similarly, we did not ask in [56] that $\nabla_v\left(\frac{1}{p}\right)\in L^1(\mathbf{R}^N)$. In particular we have to check that such a p exists. Recall that we have shown in [56] that there exists $\Phi\in C_0(\mathbf{R}^N_v)$, $\Phi>0$ on $\mathbf{R}^N$ such that $\frac{A*\phi}{1+|v|^2}\in C_0(\mathbf{R}^N)$. The construction made in [56] also shows that we may always assume: $\Phi\in L^1(\mathbf{R}^N)$. Then, let $\rho\in C^{\infty}{}_0(\mathbf{R}^N)$ be such that $\rho\geq 0$ on $\mathbf{R}^N$, $\int_{\mathbf{R}^N}\rho dv=1$. We claim that $p=(\tilde{\Phi})^{-1}$ where $\tilde{\Phi}=\Phi*\rho$ satisfies (45). Indeed, $\tilde{\Phi}>0$ on $\mathbf{R}^N$, $\nabla_v\tilde{\Phi}=\Phi*\nabla\rho\in L^1(\mathbf{R}^N)$, $\tilde{\Phi}\in C_0(\mathbf{R}^N)$ and $A*\tilde{\Phi}=(A*\Phi)*\rho$ hence $(A*\Phi)\ (1+|v|^2)^{-1}\in C_0(\mathbf{R}^N)$.

iv) In [56], we did not make any assumptions on $\nabla_v g$ in the defininion of $\mathscr{A}$. This extra assumption is needed here to take into account the force term. Many variants are possible: in particular, if, in (42), $\frac{N+2}{3}$ is replaced by a larger exponent then $\frac{N+2}{N-1}$ can be decreased to its conjugate exponent. For example, when $N=3$, $V_0=\frac{1}{|x|}$, (42) holds with $\frac{N+2}{3}=\frac{5}{3}$ replaced by 2 and thus $\frac{N+2}{N-1}=\frac{5}{2}$ can be replaced by 2. A similar observation can be made on the growth condition on β.

Of course, these remain to give a precise meaning to each of the terms between brackets in (43). This is done in Part II [56] for the first term and the last one. We do not recall the details here but we simply recall that we have on one hand

$$<\frac{\partial f}{\partial t}+v\cdot\nabla f,\,m>=<\Big(\frac{\partial}{\partial t}+v\cdot\nabla_x\Big)\Big\{\varphi\,(p\,(f-g))\frac{1}{p}\Big\},\,\varphi>$$

$$+\int_0^{+\infty}dt\int_{\mathbf{R}^N}dx\alpha'(p\,(f-g))\,\varphi\Big\{\frac{\partial g}{\partial t}+v\cdot\nabla_x g\Big\}$$

$$+<\Big(\frac{\partial}{\partial t}+v\cdot\nabla_x\Big)\beta\,(f-g),\,\psi>$$

$$+\int_0^{+\infty}dt\int_{\mathbf{R}^N}dx\beta'(f-g)\,\psi\Big\{\frac{\partial g}{\partial t}+v\cdot\nabla_x g\Big\}\;..$$

And the first and third terms make sense in distributions sense while the second and the fourth ones are meaningful in view of the assumptions made upon g. On the other hand, using the symmetries of the collision operator, we write as in [56] the term $<Q(f,f),\,m>$ as follows

$$<Q\,(f,f),\,m>=\int_0^\infty dt\int_{\mathbf{R}^N}dx\varphi\,(x,\,t)\Big\{\int_{\mathbf{R}^{2N}}dv\,dv_*\int_{S^{N-1}}d\omega$$

$$Bff_*\,\{\alpha'(p'(f'-g'))-\alpha'(p\,(f-g))\}$$

$$+\int_0^\infty dt\int_{\mathbf{R}^N}dx\int_{\mathbf{R}^{2N}}dv\,dv_*\int_{S^{N-1}}d\omega B\psi\,\{f'f'_*-ff_*\}\,\beta'(f-g)\;.$$

These two terms are shown to be meaningful in [56] because of the properties of α and β and of the "entropy dissipation bound" assumed in (A).

These remain to define "$<F\cdot\nabla_v f,\,m>$" in (43). To this end we set

$$<F\cdot\nabla_v f,\,m>=<F\cdot\nabla_v g,\,m>+\int_0^\infty dt\int_{\mathbf{R}^N}dxF\varphi\cdot\int_{\mathbf{R}^N}dv\,\nabla_v\Big(\frac{1}{p}\Big)\;.$$

$$\{p\,(f-g)\,\alpha'(p\,(f-g))-\alpha\,(p\,(f-g))\}-\int_0^\infty dt\int_{\mathbf{R}^N}F\cdot\int_{\mathbf{R}^N}dv\,\nabla_v\psi\beta\,(f-g)\;.$$

The first term makes sense since m is bounded and has compact support in $\mathbf{R}_x^N\times[0,\,\infty)$: indeed F, because of (42), belongs to $L^\infty(0,\,T;\,L_{loc}^{\frac{N+2}{3}}(\mathbf{R}_x^N))$ if $N\geq 3$, $L^\infty\,(0,\,T;\,L_{loc}^2(\mathbf{R}^2))$ if $N=2$ (for all $T\in(0,\,\infty)$) and our claim is shown in view of the integrability of $\nabla_v g$ assumed in the definition of A.

The second term is treated in a similar way since, by assumption, $p\,(f-g)$ $\alpha'(p\,(f-g))-\alpha\,(p\,(f-g))$ is bounded while $\nabla_v\frac{1}{p}\in L^1(\mathbf{R}_v^N)$. Finally, for the last term, we use the growth condition on β to deduce that

$$\int_{\mathbf{R}^N} |\nabla_v \psi||\beta(f-g)|dv \in C([0, \infty; L^1(\mathbf{R}_x^N) \cap L^\gamma(\mathbf{R}_x^N)))$$

where $\gamma = \frac{N+2}{N-1}$ if $N \geq 3$, $\gamma = 2$ if $N = 2$. In addition this integral has compact support in $\mathbf{R}_x^N \times [0, \infty)$ in view of the support of ψ. And we conclude using (42) since $F \in L^\infty(0, T; L^\infty(\mathbf{R}_x^N) + L^{\frac{N+2}{3}}(\mathbf{R}_x^N))$ if $N \geq 3$ while $F \in L^\infty\ (0, T; L^2_{loc}(\mathbf{R}_x^N))$ if $N = 2$.

Let us finally observe that the above notion of (weak) solutions is stronger than the notion of renormalized solutions introduced by R. J. DiPerna and the author [25] (for the pure Boltzmann equation) which corresponds to the special case when $m = \psi\beta'(f)$ $(\varphi \equiv \alpha \equiv g \equiv 0)$ and $\beta \in C^1([0, \infty))$, $\beta(0)$, $\beta'(t)(1+t)^{-1}$ is bounded on $[0, \infty)$. In this paper we shall say that f is a renormalized solution of (VB) if the above definition holds for multipliers m of the form $\psi\beta'(f-g)$ as in the previous definition (in other words we simply take $\psi \equiv \alpha \equiv 0$).

We may now state our main existence result recalling that assume (35), (36), (42).

Theorem II.1. *Let $f_0 \geq 0$ satisfy* (18), *then there exists a global solution f of* (VB) *satisfying* (15) *and the following entropy and energy inequalities*

$$(46) \qquad \begin{cases} \displaystyle\iint_{\mathbf{R}^{2N}} f \log f\, dx\, dv\, (t) \\ \displaystyle + \frac{1}{4}\int_0^t ds \int_{\mathbf{R}^N} dx \iint_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} B(f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*} \\ \displaystyle \qquad \leq \iint_{\mathbf{R}^{2N}} f_0 \log f_0 dx\, dv\ , \quad \textit{for all } t > 0 \end{cases}$$

and

$$(47) \qquad \begin{cases} \displaystyle\iint_{\mathbf{R}^{2N}} f(t)|v|^2 dx\, dv + \iint_{\mathbf{R}^{2N}} \rho(x,t) V_0(x-y) \rho(y,t) dx\, dy \\ \displaystyle \quad \leq \iint_{\mathbf{R}^{2N}} f_0 |v|^2 dx\, dv + \iint_{\mathbf{R}^{2N}} \rho_0(x) V_0(x-y) \rho_0(y) dx\, dy\ , \\ \qquad\qquad\qquad\qquad \textit{for all } t > 0\ . \end{cases}$$

Remark II.2. i) Since $f(\omega + |v|^2) \in L^\infty(0, T; L^1(\mathbf{R}^{2N}_{x,v}))$ for all $T \in (0, \infty)$ the left-hand side of (47) makes sense because of (35).

ii) It is not known whether equalities hold in (46) and (47) or even if the inequalities hold between $s \geq 0$ and $t \geq s$ only $s = 0$ seems to be "available".

iii) Of course, as in the case of the pure Boltzmann model, further a priori bounds or regularity properties of solutions are not known and the uniqueness is a major open problem. However, exactly as we did in Part II

[56], one can show the uniqueness of solutions provided there exists a strong one (more regular, see [56] for more details]. Our analysis is indeed readily adapted to the VB case. Concerning the regularity, let us mention that if $B\equiv 0$ i.e. the VB system reduces to the pure Vlasov system then further a priori estimates, regularity and uniqueness can be shown (if $N=3$ for example) under an assumption on V_0 which is extremely similar to (42) namely:

$$\begin{cases} \nabla V_0 \in L^\infty + L^{\frac{3}{2},\infty}(\mathbf{R}^3) \ , \\ [(D^2V_0)*] \text{ is bounded from } L^p(\mathbf{R}^3) \text{ to } L^\infty + L^p(\mathbf{R}^3) \text{ for } 1<p<\infty \ . \end{cases}$$

Indeed these conditions allow to copy the proof made in P. L. Lions and B. Perthame [57].

iv) The solution we build also satisfies

$$\iint_{\mathbf{R}^{2N}} f v_k dx\, dv \quad \text{is independent of } t\geq 0 \ , \quad \text{for } 1\leq k\leq N \ . \tag{48}$$

v) We want to emphasize the fact that the solutions built in Theorem II.1 enjoy various properities such as the local (and global) conservation of mass and a priori estimates. Since these properities are somewhat hidden in the rather complicated definition of solutions, it is worth explaining how one can recover them. First of all, we observe that choosing $\alpha(t)=t$, $\beta\equiv g\equiv 0$ we obtain from the definition of solutions

$$\frac{\partial \rho}{\partial t}+\operatorname{div}_x(j)=0 \qquad \text{in} \quad \mathscr{D}'(\mathbf{R}^N_x\times(0,\infty)) \tag{49}$$

or equivalently for all $\varphi\in C_0^\infty(\mathbf{R}^N_x\times[0,\infty))$

$$\begin{cases} \int_0^\infty dt \iint_{\mathbf{R}^{2N}} dx\, dv\, f(x,v,t)\left\{\frac{\partial\varphi}{\partial t}(x,t)+v\cdot\nabla_x\varphi(x,t)\right\} \\ \qquad\qquad +\iint_{\mathbf{R}^{2N}} dx\, dv\, f_0\varphi(x,0)=0 \ . \end{cases} \tag{49'}$$

The equality (49) is known as the local conservation of mass. Next, if we take $\varphi(x,t)=\varphi(t)\,\psi\left(\frac{x}{n}\right)$ where $\varphi\in C_0^\infty([0,\infty))$, $\psi\in C_0^\infty(\mathbf{R}^N)$, $\psi\equiv 1$ on $\{|x|\leq 1\}$, $\psi\equiv 0$ for $|x|\geq 2$, $n\geq 1$. We deduce easily letting n go to $+\infty$ and using the fact that $f(1+|v|^2)\in L^\infty(0,T;L^1(\mathbf{R}^{2N}_{x,v}))$ (for all $T\in(0,\infty)$) the following global conservation of mass

$$\iint_{\mathbf{R}^{2N}} f(t)\,dx\, dv \quad \text{is independent of} \quad t\geq 0 \ . \tag{50}$$

Similarly, using $\varphi=\varphi(t)\,\psi\left(\frac{x}{n}\right)\omega(x)$ and the bounds assumed on f, we obtain

$$\frac{\partial}{\partial t}\left(\iint_{\mathbf{R}^{2N}} f\omega\, dx\, dv\right)=\iint_{\mathbf{R}^{2N}} f(v\cdot\nabla_x\omega)\,dx\, dv \ . \tag{51}$$

Observe indeed that

$$\omega|\nabla\varphi_n(x)|\leq C\frac{1}{n}1_{n\leq|x|\leq 2n}\omega\leq C\sqrt{1+\omega}\,1_{n\leq|x|\leq 2n}$$

since $\omega\leq C(1+|x|^2)$. This is why we have

$$\int_0^\infty dt\int\int_{\mathbf{R}^{2N}}\varphi(t)f(x,v,t)v\cdot\nabla\varphi_n\omega\,dx\,dv\underset{n}{\rightarrow}0\ .$$

Then, (50), (51), (46), (47) allow to deduce exactly as we did in the beginning of this section the following a priori estimates: for all $T\in(0,\infty)$, there exists a positive constant C such that

$$(52)\quad\begin{cases}\sup_{t\in[0,T]}\left\{\int\int_{\mathbf{R}^{2N}}f(t)\ \{1+|v|^2+\omega(x)+|\log f(t)|+|V_0|*\rho(t)\}\,dx\,dv\right\}\\ +\int_0^T dt\int_{\mathbf{R}}dx\int_{\mathbf{R}^{2N}}dv dv_*\int_{S^{N-1}}d\omega B(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}\\ \qquad\leq C\left\{\int\int_{\mathbf{R}^{2N}}f_0(1+|v|^2+\omega+|\log f_0|+|V_0|*\rho_0)\,dx\,dv+1\right\}\ .\end{cases}$$

Theorem II.1 is shown in section III. Its proof relies upon some stability results we wish to state now. They concern sequences of solutions - we could consider as well approximated solutions... - with uniform natural bounds and show, extracting subsequences if necessary, the weak convergence in L^1 to a renormalized solution and the strong convergence in L^1, uniformly $t\in[0,T]$ (for all $T\in(0,\infty)$), if the initial conditions converge strongly in L^1. More precisely, we make the same assumptions as before on V_0 and B and we consider a sequence of nonnegative initial conditions $(f_0^n)_{n\geq 1}$ satisfying (20). Without loss of generality, we may assume that f_0^n converges weakly in $L^1(\mathbf{R}^{2N}_{x,v})$ to some f_0 (which then satisfies (36)). Then, we consider a sequence $(f^n)_{n\geq 1}$ of renormalized solutions of (VB) such that $f^n|_{t=0}\equiv f_0^n$ on $\mathbf{R}^{2N}$ and satisfying for all $T\in(0,\infty)$

$$(53)\quad\begin{cases}\sup_{\substack{n\geq 1\\ t\in[0,T]}}\left\{\int\int_{\mathbf{R}^{2N}}dx\,dv\,f^n(t)\ \{1+|v|^2+\omega(x)+|\log f^n(t)|+|V_0|*\rho^n(t)\}\right.\\ \quad+\sup_{n\geq 1}\int_0^T dt\int_{\mathbf{R}^N}dx\int\int_{\mathbf{R}^{2N}}dv\,dv_*\int_{S^{N-1}}d\omega\,B(f^{n\prime}f^{n\prime}_*-f^nf^n_*)\log\frac{f^{n\prime}f^{n\prime}_*}{f^nf^n_*}<\infty\ .\end{cases}$$

Of course, we denote by $\rho^n(t)=\int_{\mathbf{R}^N}f^n(x,v,t)\,dv$.

The existence of such a sequence f^n, given the sequence f_0^n, is insured by Theorem II.1. In fact Theorem II.1 (see Remark II.2) provides not only a renormalized solution satisfying (53) but a (weak) solution satisfying even more precise bounds than (53). Without loss of generality (extracting a subsequence if necessary), we may assume that f^n converges weakly in $L^p(0, T; L^1(\mathbf{R}^{2N}_{x,v}))$ (for all $1 \leq p < \infty$, $T \in (0, \infty)$) to some $f \geq 0$ which belongs to $L^\infty(0, T; L^1(\mathbf{R}^{2N}_{x,v}))$ (for all $T \in (0, \infty)$).

We may now state our main convergence results where we say that $\phi^n = \phi^n(y, t)$, $y \in \mathbf{R}^k$, $k \geq 1$, $t \in [0, \infty)$ converges locally in measure to ϕ if we have for all $R \in (0, \infty)$, $\alpha > 0$

$$\text{meas}_{y,t}\{|\phi^n(y, t) - \phi(y, t)| \geq \alpha, |y| \leq R, t \in [0, R]\} \underset{n}{\rightarrow} 0 .$$

Theorem II.2. *The weak limit f is a renormalized solution of* (VB) *and we have:*

1) *For all* $\psi \in L^\infty_{loc}(\mathbf{R}^N)$ *such that* $|\psi(v)| \leq C(1 + |v|^2)^{\alpha/2}$ *a.e. on* $\mathbf{R}^N$ *for some* $\alpha < 2$ *then* $\int_{\mathbf{R}^N} f^n \psi \, dv$ *converges to* $\int_{\mathbf{R}^N} f\psi dv$ *in* $L^p(0, T; L^1(\mathbf{R}^N_x))$ *for all* $1 \leq p < \infty$, $T \in (0, \infty)$.

2) $L(f^n)$ *converges to* $L(f)$ *in* $L^p(0, T; L^1(\mathbf{R}^N_x \times K))$ *for all* $1 \leq p < \infty$, $T \in (0, \infty)$, *K compact set in* $\mathbf{R}^N_v$.

3) *For all* $\psi \in L^\infty(\mathbf{R}^N)$ *with compact support,* $\int_{\mathbf{R}^N} Q^\pm(f^n, f^n)\psi dv$ *converges locally in measure to* $\int_{\mathbf{R}^N} Q^\pm(f, f)\psi dv$. *And* $Q^\pm(f^n, f^n) \cdot (1 + f^n)^{-1}$ *are relatively weakly compact in* $L^1(\mathbf{R}^N_x \times K \times (0, T))$ *for all* $T \in (0, \infty)$, *K compact set in* $\mathbf{R}^N_v$.

4) $Q^+(f^n, f^n)$ *converges locally in measure to* $Q^+(f, f)$.

Remark II.3. In addition, we have the following inequalities that are deduced from the above result exactly as in [26] by (essentially) convexity arguments

$$\underline{\lim}_n \int\int_{\mathbf{R}^{2N}} f^n(t) \log f^n(t) \, dx \, dv \geq \int\int_{\mathbf{R}^{2N}} f(t) \log f(t) \, dx \, dv$$

$$\underline{\lim}_n \int_0^t ds \int_{\mathbf{R}^N} dx \int\int_{\mathbf{R}^{2N}} dv \, dv_* \int_{S^{N-1}} d\omega B (f^{n\prime} f^{n\prime}_* - f^n f^n_*) \log \frac{f^{n\prime} f^{n\prime}_*}{f^n f^n_*}$$
$$\geq \int_0^t ds \int_{\mathbf{R}^N} dx \int\int_{\mathbf{R}^{2N}} dv \, dv_* \int_{S^{N-1}} d\omega B (f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*}$$

for all $t \geq 0$.

Theorem II.3. *If in addition* f^n_0 *converges in* $L^1(\mathbf{R}^{2N}_{x,v})$ *to* f_0, *then* f^n *converges to f in* $C([0, T]; L^1(\mathbf{R}^{2N}_{x,v}))$ *for all* $T \in [0, \infty)$. *And f is a solution of the* (*VB*) *system if* $(f^n)_n$ *is a sequence of solutions.*

Remark II.4. Parts 1)-3) of Theorem II.2 are the analogues of results shown in R. J. DiPerna and P. L. Lions [25] for the "pure" Boltzmann model. Part 4) corresponds to the main compactness result shown in Part I [55] while Theorem II.3 is the analogue of the convergence result shown in Part II [56] again for the Boltzmann equation.

Theorems II.2 and II.3, as Theorem II.1, are proven in the following section.

III. Proofs

We begin with the *proof of Theorem* II.2. We divide it into two main steps. In the first one, we briefly explain why parts 1) -4) hold. And we concentrate in the second step on the proof of the fact that the weak limit is indeed a renormalized solution of VB. If the first step is essentially an adaptation of the results and methods of [25] ([26]) and [52], the second one requires a new argument which may be seen as a simplification and extension of the original argument introduced in [25] for the weak passage to the limit in the context of Boltzmann equations.

Step 1. In fact, we are only going to prove 1) following the arguments of [25] (see also [52]) since 2) and 3) are then shown exactly as in [25]. Finally, once 1) -3) hold, the proof presented in Part I [56] immediately yields 4). In order to prove 1), we first recall that for all compact sets $K\subset\mathbf{R}_v^N$, $T\in(0,\infty)$. Indeed, we have

$$\iint_{\mathbf{R}^N\times K}dx\,dv\,(1+f^n)^{-1}Q^-(f^n,f^n)\leq\iint_{\mathbf{R}^N\times K}dx\,dvL^n(f)$$
$$=\int_{\mathbf{R}^N}dx\int_{\mathbf{R}^N}f^n(x,v_*,t)\int_K A(v-v_*)dv$$
$$\leq C\int_{\mathbf{R}^N}dx\int_{\mathbf{R}^N}f^n(x,v_*,t)(1+|v_*|^2)dv_*$$

in view of (7). And (54) follows from (53).

Next, we observe that we have

$$Q^+(f^n,f^n)\leq 2Q^-(f^n,f^n)$$
$$+\frac{1}{\log 2}\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}B\,d\omega\,(f^{n\prime}f_*^{n\prime}-f^nf_*^n)\log\frac{f^{n\prime}f_*^{n\prime}}{f^nf_*^n}\ .$$

Hence, (53) and (54) imply

(55) $\qquad(1+f^n)^{-1}Q^+(f^n,f^n)$ is bounded in $L^1(0,T;L^1(\mathbf{R}_x^N\times K))$

for all compact sets K in $\mathbf{R}_v^N$, $T\in(0,\infty)$.

Next, we observe that since f^n is a renormalized solution of (VB) we have for all $\beta\in C^1([0,\infty);\mathbf{R})$ such that $\beta(0)=0$, $\beta'(t)(1+t)$ is bounded on $\mathbf{R}$

(56) $$\left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\beta(f^n)=-\mathrm{div}_v\{F^n\beta(f^n)\}+\beta'(f^n)Q(f^n,f^n)\quad\text{in }\mathcal{D}'\ .$$

In order to apply the velocity averaging results of [32], we remark that (54) and (55) imply that $\beta'(f^n)Q(f^n,f^n)$ is bounded in $L^1(0,T;L^1(\mathbf{R}_x^N\times K))$ for all compact subsets K of $\mathbf{R}_v^N$, $T\in(0,\infty)$. And (53) shows that $\beta(f^n)$ is bounded in $L^\infty(0,T;L^p(\mathbf{R}_{x,v}^{2N}))$ for all $T\in(0,\infty)$, $1\le p<\infty$. Finally, we deduce from (42) that $F^n=F^n(x,t)$ is bounded in $L^\infty(0,T;L^\infty+L^{\frac{N+2}{3}}(\mathbf{R}_x^N))$, hence $F^n\beta(f^n)$ is bounded in $L^\infty(0,T;L^q_{loc}(\mathbf{R}_{x,v}^{2N}))$ for all $T\in(0,\infty)$, $1\le q<\frac{N+2}{3}$.

At this stage, we may use (56) and the velocity averaging results of [32] to deduce from those bounds

(57) $\int_{\mathbf{R}_v^N}\beta(f^n)\psi dv$ is relatively compact in $L^p_{loc}(\mathbf{R}_x^N\times(0,\infty))$ for all $1\le p<\infty$, $\psi\in L^\infty(\mathbf{R}_v^N)$ with compact support. Then, (53) yields

(58) $\int_{\mathbf{R}_v^N}\beta(f^n)\psi dv$ is relatively compact in $L^p(0,T;L^p(0,T;L^1(\mathbf{R}_x^N))$

for all $1\le p<\infty$, $T\in(0,\infty)$, $\psi\in L^\infty_{loc}(\mathbf{R}_v^N)$ such that $\psi(1+|v|^2)^{-1}$ goes to 0 as $|v|$ goes to $+\infty$.

Part 1) then follows from (58) and (53) since we only have to choose $\beta=\beta_\delta=\frac{1}{\delta}\log(1+\delta t)$ ($\delta\in(0,1]$) and to observe that we have for all $R>1$

$$0\le f^n-\beta_\delta(f^n)\le\delta Rf^n+f^n 1_{(fn>R)}\le\delta Rf^n+f^n\frac{\log f^n}{\log R}\ .$$

Step 2. We wish to prove now that f is a renormalized solution of (VB). First of all, we observe that it is enough to show that the following holds

$$\frac{\partial}{\partial t}\beta(f)+\mathrm{div}_v\{v\beta(f)\}+\mathrm{div}_v(F\beta(f))=\beta'(f)Q(f,f)\quad\text{in }\mathcal{D}'\tag{59}$$

where $\beta(t)=\log(1+t)$, $F=-\nabla V_0*\rho$, $\rho=\int_{\mathbf{R}^N}fdv$.

Indeed, it is shown in the Appendix 2 that (59) then implies that f is a renormalized solution of (VB).

Let us also recall that we deduce from (53) and weak passages to the limit

$$\begin{cases}\sup_{t\in[0,T]}\Big\{\int\int_{\mathbf{R}^{2N}}f(t)\,(1+\omega(x)+|v|^2+|\log f(t)|+|V_0|*\rho(t))dx\,dv\\ +\int_0^T ds\int_{\mathbf{R}^N}dx\int_{\mathbf{R}^{2N}}dv\,dv_*\int_{S^{N-1}}Bd\omega\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}\Big\}<\infty\end{cases}\tag{60}$$

for all $T\in(0,\infty)$.

Let us now explain the strategy of proof we use to derive (59). We consider $\beta_\delta(f^n)=f^n(1+\delta f^n)^{-1}$ for $\delta\in(0,1]$ and weakly pass to the limit as n goes to $+\infty$ in the equation satisfied by $\beta_\delta(f^n)$ (we use here the fact that f^n

is a renormalized solution). Then, we renormalize (taking β of the limit of $\beta_\delta(f^n)$) the resulting limit equation and we let δ go to 0_+ to recover (59).

In order to do so, we need a few notations. Without loss of generality, extracting subsequence if necessary, we may assume that for all $\delta>0$

(61) $$\beta_\delta(f^n) \underset{n}{\rightarrow} \beta_\delta \quad \text{weakly in} \quad L^p(\mathbf{R}^{2N}_{x,v}\times(0,T))$$

(62) $$h^n_\delta=(1+\delta f^n)^{-2} \underset{n}{\rightarrow} h_\delta \quad \text{weakly in} \quad L^\infty(\mathbf{R}^{2N}_{x,v}\times(0,\infty)) \ (\text{weak}*)$$

(63) $$g^n_\delta=f^n(1+\delta f^n)^{-2} \underset{n}{\rightarrow} g_\delta \quad \text{weakly in} \quad L^p(\mathbf{R}^{2N}_{x,v}\times(0,T))$$

for all $T\in(0,\infty)$, $1\leq p\leq\infty$ ($p=\infty$ means the weak $*$ convergence...). Furthermore, because of part 3), we may assume that

(64) $$(1+\delta f^n)^{-2}Q^\pm(f^n,f^n) \underset{n}{\rightarrow} Q^\pm_\delta \quad \text{weakly in} \quad L^1(\mathbf{R}^N_x\times K\times(0,T))$$

for all compact sets $K\subset\mathbf{R}^N_v$, $T\in(0,\infty)$.

Of course, (56) holds with β replaced by β_δ for all $\delta>0$ and we want to pass to the limit in these equations as n goes to $+\infty$. To this end, we deduce from part I that ρ^n converges in $L^p(0,T;L^1(\mathbf{R}^N_x))$ (for all $1\leq p<\infty$, $T\in(0,\infty)$) to ρ. Since $\nabla V_0\in L^1_{loc}(\mathbf{R}^N)$, we deduce that F^n converges in $L^p(0,T;L^1_{loc}(\mathbf{R}^N))$ to $F=-\nabla V_0*\rho$. And finally (42) yields

(65) $$\begin{cases} F^n \underset{n}{\rightarrow} F \quad \text{in} \quad L^p(0,T;L^q_{loc}(\mathbf{R}^N)) \\ \qquad \text{for all } 1\leq p<\infty,\ 1\leq q<\dfrac{N+2}{3} \end{cases}$$

and for all $T\in(0,\infty)$.

We then pass to the limit in (56) and we obtain

(66) $$\frac{\partial}{\partial t}\beta_\delta+\operatorname{div}_x(v\beta_\delta)+\operatorname{div}_v(F\beta_\delta)=Q^+_\delta-Q^-_\delta \quad \text{in} \quad \mathcal{D}'.$$

Next, we observe that the vector-field $B=(v,F(t,x))$ satisfies: $\operatorname{div}_{x,v}(B)=0$ and $B\in L^\infty(0,T;W^{1,1}_{loc}(\mathbf{R}^{2N}_{x,v}))$. We may thus apply the general results of R. J. DiPerna and P. L. Lions [29] on linear transport equations and ODE's to deduce that β_δ is a renormalized solution of (66).

This fact has many consequences, one of which is the continuity of β_δ with respect to $t\geq 0$ with values in $L^p(\mathbf{R}^{2N}_{x,v})$ for all $1\leq p<\infty$. In view of (53), it is clearly enough to show that $\beta_\delta\in C([0,\infty);L^p_{loc}(\mathbf{R}^{2N}_{x,v}))$ ($\forall\, 1\leq p<\infty$). To this end, we remark that if we regularize by convolution (with a mollifier) β_δ into β^ε_δ as in [29], we obtain

$$\frac{\partial}{\partial t}\beta_\delta^\varepsilon+v\cdot\nabla_x\beta_\delta^\varepsilon+F\cdot\nabla_v\beta_\delta^\varepsilon=Q_\delta^+-Q_\delta^-+r^\varepsilon$$

where $r^\varepsilon\to 0$ (as ε goes to 0) in $L^1(0, T; L^1_{loc}(\mathbf{R}^{2N}_{x,v}))$ for all $T\in(0,\infty)$. In addition, $\beta_\delta^\varepsilon\in C([0,\infty); L^p(\mathbf{R}^{2N}_{x,v}))$ for $1\le p<\infty$ because of the equation it satisfies. Next, we deduce from (66) and the fact β_δ is a renormalized solution of (66)

$$\frac{d}{dt}\int_{\mathbf{R}^{2N}}|\beta_\delta-\beta_\delta^\varepsilon|^p\varphi\,dxdv\to 0 \qquad \text{in} \quad L^1(0,T) \;, \quad \text{as} \quad \varepsilon\to 0_+$$

for all $1\le p<\infty$, $T\in(0,\infty)$, $\varphi\in C_0^\infty(\mathbf{R}^{2N}_{x,v})$, $\varphi\ge 0$. And we conclude.

This strong continuity in t allows to deduce that necessarily $f\in C([0,\infty); L^1(\mathbf{R}^{2N}_{x,v}))$. Indeed, because of (53) - see step 1 for a similar argument-, we have for all $T\in(0,\infty)$

$$\sup_{n\ge 1}\ \sup_{t\in[0,T]}\ \|f^n-\beta_\delta(f^n)\|_{L^1(\mathbf{R}^{2N}_{x,v})}\to 0 \qquad \text{as} \quad \delta\to 0_+ \;. \tag{67}$$

Hence, β_δ converges in $C([0,T]; L^1(\mathbf{R}^{2N}_{x,v}))$ to f proving thus our claim.

We next wish to precise (66). In order to do so, we observe that $-\frac{t}{1+\delta t}$, $\frac{1}{(1+\delta t)^2}$ are convex on $[0,\infty)$ therefore we have

$$\beta_\delta\le\beta_\delta(f) \;, \quad h_\delta\ge(1+\delta f)^{-2} \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v}\times(0,\infty) \;. \tag{68}$$

In addition $\frac{t}{(1-\delta t)^2}=\beta_\delta(t)\,(1-\delta\beta_\delta(t))$, hence

$$g_\delta\le\beta_\delta(1-\delta\beta_\delta) \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v}\times(0,\infty) \;. \tag{69}$$

Furthermore, because of part 2 and (62), we deduce easily

$$Q_\delta^-=g_\delta L(f) \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v}\times(0,\infty) \;. \tag{70}$$

In fact, using part 4, we could also deduce that

$$Q_\delta^+=h_\delta Q^+(f,f) \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v}\times(0,\infty) \;. \tag{71}$$

But since (71) is not needed for our argument here (and thus part 4 is not needed here) we shall only prove this property in the course of proving Theorem II.3.

We finally use the fact that β_δ is a renormalized solution of (66) to write

$$\begin{cases} \frac{\partial}{\partial t}(\beta(\beta_\delta))+\operatorname{div}_x\{v\beta(\beta_\delta)\}+\operatorname{div}_v\{F\beta(\beta_\delta)\} \\ \qquad =(1+\beta_\delta)^{-1}Q_\delta^+-(1+\beta_\delta)^{-1}Q_\delta^- \;. \end{cases} \tag{72}$$

And we wish to recover (59) letting δ go to 0_+. Recall that we already saw above that β_δ converges to f in $C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ for all $T\in(0,\infty)$. Therefore, in order to complete the proof of Theorem II.2, there only remains to show

$$(73)\qquad \begin{cases}(1+\beta_\delta)^{-1}Q_\delta^-\to(1+f)^{-1}Q^-(f,f)\ ,\\ (1+\beta_\delta)^{-1}Q_\delta^+\to(1+f)^{-1}Q^+(f,f)\qquad \text{a.e.}\end{cases}$$

as δ goes to 0_+, and

$$(74)\qquad Q_\delta^\pm(1+\beta_\delta)^{-1}\ \text{ are weakly relatively compact in } L^1(\mathbf{R}^N_x\times K\times(0,T))$$

for all compact sets $K\subset\mathbf{R}^N_v$, $T\in(0,\infty)$.

We begin with Q_δ^-. Without loss of generality, we may assume that β_δ converges a.e. to f as δ goes to 0_+. Then,

$$(1+\beta_\delta)^{-1}Q_\delta^-=(1+\beta_\delta)^{-1}g_\delta L(f)\underset{\delta}{\to}(1+f)^{-1}fL(f)\quad \text{a.e.}$$

provided we show that g_δ converges a.e. to f.

This is easy since we have for all $R>1$

$$0\le f^n-f^n(1+\delta f^n)^{-2}\le 3R\delta f^n+f^n 1_{(fn>R)}$$

hence g_δ converges to f in $C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ $(\forall\, T\in(0,\infty))$. We now prove (74) for Q_δ^- and we simply observe that (69) yields

$$\begin{aligned}0&\le(1+\beta_\delta)^{-1}Q_\delta^-=(1+\beta_\delta)^{-1}g_\delta L(f)\\ &\le(1-\delta\beta_\delta)\frac{\beta_\delta}{1+\beta_\delta}L(f)\le L(f)\quad \text{a.e.}\end{aligned}$$

And we conclude since $L(f)\in L^\infty(0,T;L^1(\mathbf{R}^N_x\times K))$ for all compact sets $K\subset\mathbf{R}^N_v$, $T\in(0,\infty)$.

We conclude the proof of Theorem II.2 by proving (73) and (74) for Q_δ^+. We begin with (74). And we recall the following classical inequality valid for all $K>1$

$$(75)\qquad Q^+(f^n,f^n)\le KQ^-(f^n,f^n)+\frac{1}{\log K}e^n$$

where $e^n=\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}Bd\omega\,(f^{n\prime}f^{n\prime}_*-f^nf^n_*)\log\dfrac{f^{n\prime}f^{n\prime}_*}{f^nf^n_*}$ is bounded in $L^1_+(\mathbf{R}^{2N}_{x,v}\times(0,T))$ for all $T\in(0,\infty)$. Without loss of generality, we may assume that e^n converges weakly in the sense of measures to some bounded nonnegative measure e on $\mathbf{R}^{2N}_{x,v}\times[0,\infty)$ and we denote by e_0 its regular part $(e_0=\frac{De}{Dy})$. Dividing (75) by $(1+\delta f^n)$ and letting n go to $+\infty$, we obtain

(76) $$Q_\delta^+ \leq K Q_\delta^- + \frac{1}{\log K} e$$

hence

(77) $$Q_\delta^+ \leq K Q_\delta^- + \frac{1}{\log K} e_0 \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v} \times (0, \infty) .$$

And (74) is proven for Q_δ^+ since we already showed it for Q_δ^-.

We finally prove (73) for Q_δ^+. We first remark that we have for all $R>0$

(78) $$\begin{cases} Q^+(f^n, f^n) \geq (1+\delta f^n)^{-2} Q^+(f^n, f^n) \\ \qquad\qquad \geq (1+\delta R)^{-2} Q^+(f^n, f^n) 1_{(f^n<R)} . \end{cases}$$

In particular, if we multiply (78) by $\psi \in C_0^\infty(\mathbf{R}^N_v)$, $\psi \geq 0$, we find letting n go to $+\infty$ and using part 3

$$\int_{\mathbf{R}^N} Q^+(f, f) \psi dv \geq \int_{\mathbf{R}^N} Q_\delta^+ \psi dv \qquad \text{a.e. on} \quad \mathbf{R}^N_x \times (0, \infty) .$$

Indeed the integrated left-hand side converges locally in measure while the right-hand side converges weakly in L^1 and this is enough to pass to the limit in the a.e. inequality on $\mathbf{R}^N_x \times (0, \infty)$. Therefore, we have for all $\delta \in (0, 1]$

(79) $$Q^+(f, f) \geq Q_\delta^+ .$$

Next, we use the other part of the inequality (78) and we write for $\nu \in (0, 1]$ using (75)

(80) $$\begin{cases} (1+\delta R)^{-2} (1+\nu L(f^n))^{-1} Q^+(f^n, f^n) \\ \leq (1+\delta f^n)^{-2} Q^+(f^n, f^n) + (1+\nu L(f^n))^{-1} 1_{(f^n>R)} Q^+(f^n, f^n) \\ \leq (1+\delta f^n)^{-2} Q^+(f^n, f^n) + \frac{1}{\log K} e^n + \frac{K}{\nu} f^n 1_{(f^n>R)} . \end{cases}$$

We then observe that $Q^+(f^n, f^n)(1+\nu L(f^n))^{-1}$ is relatively weakly compact in $L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$ $(\forall T \in (0, \infty))$ since it is bounded by $\frac{1}{\log K} e^n + K f^n$ for all $K > 1$. Hence, we may assume without loss of generality that it converges weakly in $L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$ $(\forall T \in (0, \infty))$. We claim its weak limit is given by $(1+\nu L(f))^{-1} Q^+(f, f)$. Indeed, if $\psi \in L^\infty(\mathbf{R}^N_v)$ with compact support, we have

$$\int_{\mathbf{R}^N} (1+\nu L(f^n))^{-1} Q^+(f^n, f^n) \psi dv = \int_{\mathbf{R}^N} Q^+(f^n, f^n) \psi^n_\nu dv$$

where ψ^n_ν is uniformly bounded in $L^\infty(\mathbf{R}^N_v)$, has a uniform compact support and $\psi^n_\nu \underset{n}{\rightarrow} \psi_\nu = (1+\nu L(f))^{-1} \psi$ in $L^p(\mathbf{R}^N_v)$ $(\forall\, 1 \leq p < \infty)$. Their properties are enough to adapt the proof of part 3) (see [26], [27] for related arguments) and to deduce

$$\int_{\mathbf{R}^N} Q^+(f^n, f)^n \psi_\nu^n dv \underset{n}{\rightarrow} \int_{\mathbf{R}^N} Q^+(f, f)\psi_\nu \, dv$$

locally in measure on $\mathbf{R}_x^N \times [0, \infty)$. And our claim is shown.

We then pass to the limit in (80) and deduce as above

$$(81) \qquad (1+\delta R)^{-2}(1+\nu L(f))^{-1} Q^+(f, f) \leq Q_\delta^+ + \frac{1}{\log K} e_0 + \frac{K}{\nu} f_R \quad \text{a.e.},$$

where f_R is the weak limit of $f^n 1_{(f^n > R)}$. Since we have because of (53)

$$\iint_{\mathbf{R}^{2N}} dx\, dv\, f_R = \lim_n \iint_{\mathbf{R}^{2N}} dx\, dv\, f^n 1_{(f^n > R)} \leq \frac{C}{\log R},$$

we deduce from (81) letting first δ go to 0_+, then K go to $+\infty$, then R go to $+\infty$ and finally ν go to 0_+

$$(82) \qquad Q^+(f, f) \leq \lim_{\delta \to 0_+} Q_\delta^+ \quad \text{a.e.}$$

The combination of (79) and (82) completes the proof of (73) and thus of Theorem II.2.

We now turn to *the proof of Theorem* II.3.

We keep the notations of the previous proof. And we know that $f \in C([0, \infty); L^1(\mathbf{R}_{x,v}^{2N}))$ is a renormalized solution of (VB). In particular, we know that we have, setting $\gamma_\delta(t) = \frac{1}{\delta}\log(1+\delta t)$

$$(83) \qquad \begin{cases} \dfrac{\partial}{\partial t}(\gamma_\delta(f)) + \operatorname{div}_x \{v\gamma_\delta(f)\} + \operatorname{div}\{F\gamma_\delta(f)\} \\ \qquad = \gamma'_\delta(f) Q^+(f, f) - \{f\gamma'_\delta(f)\} L(f) \qquad \text{in } \mathscr{D}'. \end{cases}$$

Of course, $\gamma_\delta(f) \in C([0, \infty); L^p(\mathbf{R}_{x,v}^{2N}))$ for all $1 \leq p < \infty$ and

$$(84) \qquad \gamma_\delta(f)|_{t=0} = \gamma_\delta(f_0) \qquad \text{a.e. on } \mathbf{R}_{x,v}^{2N}.$$

We are going to follow the scheme of the first proof of the analogous result shown in Part II [56] for the "pure" Boltzmann's equation. Once more the force term introduces specific difficulties. Let us first explain the strategy of proof. We introduce, without loss of generality, the weak limit of $\gamma_\delta(f^n)$ (in $L^p(\mathbf{R}_{x,v}^{2N} \times (0, T))$ for all $T \in (0, \infty)$, $1 \leq p < \infty$) that we denote by γ_δ. The first step consists in showing that γ_δ is a supersolution of (83). In a second step, we deduce that $\gamma_\delta \equiv \gamma_\delta(f)$ and that f^n converges to f a.e. or in $L^1(\mathbf{R}_{x,v}^{2N} \times (0, T))$ (for all $T \in (0, \infty)$). Finally (step 3), we show that f^n converges to f in $C([0, T]; L^1(\mathbf{R}_{x,v}^{2N}))$ $(\forall T \in (0, \infty))$ proving thus Theorem II.3.

Before we begin this proof, we wish to make a few preliminary remarks.

Adapting the proof made above (step 2), we show that γ_δ satisfies: $\gamma_\delta \in L^\infty(0, T; L^p(\mathbf{R}^{2N}_{x,v}))$ $(\forall T \in (0, \infty), \forall 1 \leq p < \infty)$

(85) $$0 \leq \gamma_\delta \leq \gamma_\delta(f) \quad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v} \times (0, \infty)$$

and

(86) $$\frac{\partial \gamma_\delta}{\partial t} + \operatorname{div}_v \{v\gamma_\delta\} + \operatorname{div}_v \{F\gamma_\delta\} = Q^+_\delta - Q^-_\delta \quad \text{in} \quad \mathcal{D}'$$

where Q^+_δ, Q^-_δ are respectively the weak limits (in $L^1(\mathbf{R}^N_x \times K \times (0, T))$ for all compact sets $K \subset \mathbf{R}^N_v$, $T \in (0, \infty)$) of $(1+\delta f^n)^{-1} Q^+(f^n, f^n)$, $(1+\delta f^n)^{-1} Q^-(f^n, f^n)$. In fact, we claim that γ_δ is a renormalized solution of (86) and belongs to $C([0, \infty); L^p(\mathbf{R}^{2N}_{x,v}))$ $(\forall 1 \leq p < \infty)$. The proof made above almost adapts to this case except for some minor technicality due to the fact that γ_δ is no more bounded. This difficulty is circumvented by introducing $\gamma_\delta(\beta_\varepsilon(f^n))$ for $\varepsilon \in (0, 1]$ and its weak limit denoted by $\gamma^\varepsilon_\delta$. Then, the proof made in Step 2 above adapts and shows that $\gamma^\varepsilon_\delta \in C([0, \infty); L^p(\mathbf{R}^N_{x,v}))$ $(1 \leq p < \infty)$ is a renormalized solution of

(87) $$\frac{\partial}{\partial t} \gamma^\varepsilon_\delta + \operatorname{div}_x \{v\gamma^\varepsilon_\delta\} + \operatorname{div}_v \{F\gamma^\varepsilon_\delta\} = Q^+_{\delta,\varepsilon} - Q^-_{\delta,\varepsilon} .$$

The proof of our claim follows upon letting ε go to 0_+ since we have

$$0 \leq \gamma_\delta(f^n) - \gamma^\varepsilon_\delta(f^n) \leq f^n - \beta_\varepsilon(f^n) \to 0 \quad \text{in} \quad L^1(\mathbf{R}^{2N}_{x,v})$$

uniformly in $n \geq 1$, $t \in [0, T]$ $(\forall T \in (0, \infty))$,

$$0 \leq (\gamma'_\delta(f^n) - \gamma'_\delta(\beta_\varepsilon(f^n))\beta'_\varepsilon(f^n)) Q^-(f^n, f^n)$$
$$\leq \frac{C}{\delta^2}\left(\varepsilon + \frac{\varepsilon f^n}{1+\varepsilon f^n}\right) L(f^n) \xrightarrow{\varepsilon} 0 \quad \text{in} \quad L^1(\mathbf{R}^N_x \times K)$$

uniformly in $n \geq 1$, $t \in [0, T]$ for all $T \in (0, \infty)$, compact sets $K \subset \mathbf{R}^N_v$, and

$$0 \leq (\gamma'_\delta(f^n) - \gamma'_\delta(\beta_\varepsilon(f^n))\beta'_\varepsilon(f^n)) Q^+(f^n, f^n)$$
$$\leq K(\gamma'_\delta(f^n) - \gamma'_\delta(\beta_\varepsilon(f^n))\beta'_\varepsilon(f^n)) Q^-(f^n, f^n) + \frac{1}{\log K} e^n ,$$

for all $K > 1$.

Step 1. γ_δ is a supersolution of (86)

Without loss of generality, we may assume that we have

$$\gamma'_\delta(f^n) = \frac{1}{1+\delta f^n} \xrightarrow[n]{} \zeta\delta \quad \text{weakly } (*) \text{ in } \quad L^\infty(\mathbf{R}^{2N}_{x,v} \times (0, \infty))$$

and

$$f^n \gamma'_\delta(f^n) = \frac{f^n}{1+\delta f^n} \xrightarrow[n]{} \xi_\delta \quad \text{weakly } (*) \text{ in } \quad L^\infty(\mathbf{R}^{2N}_{x,v} \times (0, \infty)) .$$

Furthermore, since $\gamma'_\delta(t)$, $-t\gamma'_t(t)$ are convex on $[0, \infty)$, we deduce the following inequalities

$$(88)\qquad \begin{cases} \zeta_\delta \geq \dfrac{1}{1+\delta f} = \gamma'_\delta(f)\ , \quad \xi_\delta \leq \dfrac{f}{1+\delta f} = f\gamma'_\delta(f)\ , \\ \gamma_\delta \leq \dfrac{1}{\delta}\log(1+\delta f) = \gamma_\delta(f) \qquad \text{a.e. in} \quad \mathbf{R}^{2N}_{x,v} \times (0,\ \infty)\ . \end{cases}$$

In view of part 2) of Theorem II.2, we see that

$$(89)\qquad Q_\delta^- = \xi_\delta L(f) \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v} \times (0,\ \infty)\ .$$

We then wish to use part 4) of Theorem II.2 to deduce

$$(90)\qquad Q_\delta^+ = \zeta_\delta Q^+(f, f) \qquad \text{a.e. on} \quad \mathbf{R}^{2N}_{x,v} \times (0,\ \infty)\ .$$

Indeed, let C be an arbitrary compact subset of $\mathbf{R}^{2N}_{x,v} \times [0,\ \infty)$. By the Egorov's theorem, we find, for all $\varepsilon > 0$, a measurable set E such that $\text{meas}_{x,v,t}(E) \leq \varepsilon$, $Q^+(f^n, f^n)$ converges uniformly to $Q^+(f, f)$ on E^c and $Q^+(f, f)$ is integrable on E^c. Then, for all $\varphi \in L^\infty(\mathbf{R}^{2N}_{x,v} \times (0,\ \infty))$ supported in C, we deduce

$$\left| \int \varphi\ \{\gamma'_\delta(f^n) Q^+(f^n, f^n) - \zeta_\delta Q^+(f, f)\}\ dx\ dv\ dt \right|$$

$$\leq \|\varphi\|_{L^\infty} \int_E \gamma'_\delta(f^n) Q^+(f^n, f^n) + \zeta_\delta Q^+(f, f) dx\ dv\ dt$$

$$+ \int_{E^c} \varphi\ \{\gamma'_\delta(f^n) - \zeta_\delta\}\ Q^+(f, f) dx\ dv\ dt$$

$$+ \|\varphi\|_{L^\infty} \sup_{E^c} |Q^+(f^n, f^n) - Q^+(f, f)|\ .$$

The third term goes to 0 as n goes to $+\infty$, for each $\varepsilon > 0$. And so does the second term since $\varphi 1_{E^c} Q^+(f, f) \in L^1(\mathbf{R}^{2N}_{x,v} \times (0,\ \infty))$. Finally, since - see part 3) of Theorem II.2 - $\gamma'_\delta(f^n) Q^+(f^n, f^n)$ is weakly relatively compact in $L^1(\mathbf{R}^N_x \times K \times (0,\ T))$ for all compact sets $K \subset \mathbf{R}^N_v$, $T \in (0,\ \infty)$, the first term can be made arbitrarily small uniformly in n if we let ε go to 0_+. Notice also that $\zeta_\delta Q^+(f, f) \in L^1(\mathbf{R}^N_x \times K \times (0,\ T))$ since $\zeta_\delta (Q^+(f, f) \wedge R)$ is easily seen to be the weak limit of $\dfrac{1}{1+\zeta f^n}(Q^+(f^n, f^n) \wedge R)$. And this shows (90).

Then, if we collect (88), (89) and (90) and insert these informations in (86), we find

$$(91)\qquad \frac{\partial \gamma_\delta}{\partial t} + \text{div}_x\ \{v\gamma_\delta\} + \text{div}_v\ \{F\gamma_\delta\} \geq \gamma'_\delta(f) Q(f, f) \qquad \text{in} \quad \mathcal{D}'\ .$$

This inequality holds in the sense of distributions on $\mathbf{R}^{2N}_{x,v} \times (0,\ \infty)$ and in fact, it also holds in renormalized sense i.e. the $\mathcal{D}'$ inequality still holds if we replace γ_δ by $\beta(\gamma_\delta)$ where $\beta \in C^1([0,\ \infty);\ \mathbf{R})$ is nondecreasing and then we replace $\gamma'_\delta(f)Q(f,f)$ by $\beta'(\gamma_\delta)\gamma'_\delta(f)Q(f,f)$.

We conclude this first step proving that γ_δ satisfies (84). Indeed, in view of the equation satisfied by $\gamma_\delta(f^n)$, we deduce easily that $\gamma_\delta(f^n)$ converges in $C([0,T];W_{loc}^{-s,1}(\mathbf{R}^{2N}_{x,v}))$ ($\forall\, s>1$, $\forall\, T\in(0,\infty)$) to γ_δ. But, by assumption, $\gamma_\delta(f^n)|_{t=0}=\gamma_\delta(f_0^n)$ converges in $L^1(\mathbf{R}^{2N}_{x,v})$ and thus in $W_{loc}^{-s,1}(\mathbf{R}^{2N}_{x,v})$ to $\gamma_\delta(f_0)$. And, thus, we conclude that γ_δ satisfies the initial condition (84).

Step 2. $\gamma_\delta=\gamma_\delta(f)$ and f^n converges in L^1 to f.

We consider $\gamma_\delta(f)-\gamma_\delta=r_\delta\in C([0,\infty);L^p(\mathbf{R}^{2N}_{x,v}))$ ($\forall\, 1\le p<\infty$) and we observe that r_δ satisfies in view of (83) and (91)

$$\frac{\partial}{\partial t}r_\delta+\operatorname{div}_x\{vr_\delta\}+\operatorname{div}_v\{Fr_\delta\}\le 0 \quad \text{in} \quad \mathscr{D}' . \tag{92}$$

Not only this inequality holds in the sense of distributions but it holds in renormalized sense in view of the "renormalized calculus" established in R. J. DiPerna and P. L. Lions [29]. Furthermore, in view of step 1, we know that we have

$$r_\delta\ge 0 \quad \text{a.e. on } \mathbf{R}^{2N}_{x,v}\times(0,\infty) \ , \quad r_\delta|_{t=0}=0 \quad \text{a.e. on } \mathbf{R}^{2N}_{x,v} . \tag{93}$$

We first want to prove that $r_\delta\equiv 0$. In order to do so, at least formally, we only have to integrate (92) over $\mathbf{R}^{2N}$ to find

$$\frac{d}{dt}\int_{\mathbf{R}^{2N}} r_\delta\, dx\, dv\le 0 \quad \text{in} \quad \mathscr{D}'(0,\infty) .$$

And this inequality combined with (93) yields: $r_\delta\equiv 0$ on $\mathbf{R}^{2N}_{x,v}\times(0,\infty)$.

We thus have to justify the integration over $\mathbf{R}^{2N}$. In order to do so we consider $\varphi\in C_0^\infty(\mathbf{R}^N)$, $\varphi(z)\equiv 1$ if $|z|\le 1$, $\varphi(z)\equiv 0$ if $|z|\ge 2$ and we wish to multiply (92) by $\varphi\left(\frac{x}{n}\right)\varphi\left(\frac{v}{n}\right)$. But before we do so, we remark that $\beta_\varepsilon(r_\delta)=\frac{r_\delta}{1+\varepsilon r_\delta}$ also satisfies (92) and (93) (and $\beta_\varepsilon(r_\delta)\in C([0,\infty);L^p(\mathbf{R}^{2N}_{x,v}))$ for $1\le p<\infty$) for $\varepsilon>0$. And we find finally integrating over $\mathbf{R}^{2N}\times(0,T)$ for all $t\ge 0$

$$\left\{\begin{aligned} &\int\int_{\mathbf{R}^{2N}}\beta_\varepsilon(r_\delta)\,\varphi\left(\frac{x}{n}\right)\varphi\left(\frac{v}{n}\right)dx\,dv\,(t)\le\int_0^t ds\int\int_{\mathbf{R}^{2N}}dx\,dv\,\beta_\varepsilon(r_\delta)\\ &\quad\cdot\left\{\frac{v}{n}\cdot\nabla\varphi\left(\frac{x}{n}\right)\cdot\varphi\left(\frac{v}{n}\right)+\frac{1}{n}F(x,t)\cdot\nabla\varphi\left(\frac{v}{n}\right)\varphi\left(\frac{x}{n}\right)\right\} .\end{aligned}\right. \tag{94}$$

We then wish to let n go to $+\infty$. In order to do so, we recall that (53) and (60) yield

$$\sup\left\{\int\int_{\mathbf{R}^{2N}}\beta_\varepsilon(r_\delta)|v|^2dx\,dv;\ t\in[0,T],\ \varepsilon\ge 0,\ \delta\ge 0\right\}<\infty \tag{95}$$

for all $T\in(0,\infty)$. Hence, we have the following estimates

$$\int_0^t ds\int\int_{\mathbf{R}^{2N}} dx\,dv\beta_\varepsilon(r_\delta)\left|\frac{v}{n}\cdot\nabla\varphi\left(\frac{x}{n}\right)\right|\varphi\left(\frac{v}{n}\right)$$

$$\leq\int_0^t ds\int\int_{\mathbf{R}^{2N}}\beta_\varepsilon(r_\delta)1_{(n\leq|x|\leq 2n)}2\|\varphi\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}dx\,dv\underset{n}{\longrightarrow}0\ ,$$

and

$$\int_0^t ds\int\int_{\mathbf{R}^{2N}} dx\,dv\,\beta_\varepsilon(r_\delta)\frac{1}{n}\varphi\left(\frac{x}{n}\right)\left|F(x,t)\cdot\nabla\varphi\left(\frac{v}{n}\right)\right|$$

$$\leq\left(\int_0^t ds\int\int_{\mathbf{R}^{2N}} dx\,dv\,\beta_\varepsilon(r_\delta)\frac{1}{n}|F(x,t)|1_{(n\leq|v|\leq 2n)}dx\,dv\right)\cdot\|\varphi\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}\ .$$

Then, we recall that, because of (42), $F\in L^\infty(0,t;L^\infty(\mathbf{R}_x^N)+L^{\frac{N+2}{3}}(\mathbf{R}_x^N))$. And the bounded part of F clearly gives a contribution to the preceding integral that vanishes as n goes to $+\infty$. Therefore, we may assume without loss of generality that $F\in L^\infty(0,t;L^{\frac{N+2}{3}}(\mathbf{R}_x^N))$. We next observe that because of (95)

$$\left\|\int_{\mathbf{R}^N} dv\beta_\varepsilon(r_\delta)\frac{1}{n}1_{(n\leq|v|\leq 2n)}\right\|_{L^1(0,t;L^1(\mathbf{R}_x^N))}=\frac{1}{n^3}\varepsilon_n\quad\text{with } \varepsilon_n\underset{n}{\longrightarrow}0$$

while of course we have for some $C_\varepsilon>0$

$$\left\|\int_{\mathbf{R}^N} dv\beta_\varepsilon(r_\delta)\frac{1}{n}1_{(n\leq|v|\leq 2n)}\right\|_{L^1(0,t;L^\infty(\mathbf{R}_x^N))}\leq C_\varepsilon n^{N-1}\ .$$

Therefore, we deduce from Hölder's inquality that we have for all $\varepsilon>0$

$$\int_{\mathbf{R}^N} dv\beta_\varepsilon(r_\delta)\frac{1}{n}1_{(n\leq|v|\leq 2n)}\underset{n}{\longrightarrow}0\qquad\text{in}\quad L^1(0,t;L^{\frac{N+2}{N-1}}(\mathbf{R}_x^N))\ .$$

Since $F\in L^\infty(0,t;L^{\frac{N+2}{3}}(\mathbf{R}_x^N))$, we have thus proven

$$\int_0^t ds\int\int_{\mathbf{R}^{2N}} dx\,dv\,\beta_\varepsilon(r_\delta)\frac{1}{n}|F(x,t)|1_{(n\leq|v|\leq 2n)}\underset{n}{\longrightarrow}0\ .$$

And letting first n go to $+\infty$ and then ε go to 0_+ in (94), we deduce

$$\int_{\mathbf{R}^{2N}} r_\delta(x,v,t)\,dx\,dv\leq 0\qquad\text{for all}\quad t\geq 0\ .$$

This, combined with (93), implies that $r_\delta\equiv 0$ on $\mathbf{R}_{x,v}^{2N}\times(0,\infty)$.

In other words, $\gamma_\delta(f^n)$ weakly converges to $\gamma_\delta(f)$ and, since γ_δ is strictly concave on $[0,\infty)$, we deduce from classical functional analysis arguments that f^n converges in measure to f on $\mathbf{R}_{x,v}^{2N}\times(0,T)$ for all $T\in(0,\infty)$ (observe for instance that $\gamma_\delta\left(\frac{f^n+f}{2}\right)-\frac{1}{2}(\gamma_\delta(f^n)+\gamma_\delta(f))\to 0$ in $L^1(\mathbf{R}_{x,v}^{2N}\times(0,T))$...). And this convergence, because of (53), implies

(96) $$f^n \underset{n}{\rightarrow} f \quad \text{in } L^p(0, T; L^1(\mathbf{R}^{2N}_{x,v}))\ , \quad \text{for all } 1 \le p < \infty,\ T \in (0, \infty)\ .$$

Step 3. Conclusion.

We only have to show that f^n converges to f in $C([0, T]; L^1(\mathbf{R}^{2N}_{x,v}))$ using (96) and the equation satisfied by f^n. Then, because of (53) and (67), it is clearly enough to show that, for each $\delta > 0$, $T \in (0, \infty)$, K compact set in $\mathbf{R}^{2N}_{x,v}$, we have

(97) $$\beta_\delta(f^n) \underset{n}{\rightarrow} \beta_\delta(f) \qquad \text{in} \quad C([0, T]; L^1(K))\ .$$

We then take $\varphi \in C_0^\infty(\mathbf{R}^{2N}_{x,v})$ such that $\varphi \equiv 1$ on K, $\varphi \ge 0$ and we use (56) to deduce for all $t \ge 0$

(98) $$\begin{cases} \left(\int\int_{\mathbf{R}^{2N}} \beta_\delta(f^n)^2 \varphi\, dx\, dv\right)(t) = \int_0^t ds \int\int_{\mathbf{R}^{2N}} dx\, dv \dfrac{2\beta_\delta(f^n)}{1+\delta f^n} \\ \qquad \cdot Q(f^n, f^n)\varphi + \beta_\delta(f^n)^2 \{v \cdot \nabla_x \varphi + F^n \cdot \nabla_v \varphi\}\ . \end{cases}$$

Then, because of (96), $\beta_\delta(f^n)$ converges to $\beta_\delta(f)$ in $L^p(\mathbf{R}^{2N}_{x,v} \times (0, T))$ for all $1 \le p < \infty$, $T \in (0, \infty)$ and one checks easily that the right-hand side of (98) converges uniformly in $t \in [0, T]$ ($\forall\, T \in (0, \infty)$) to the same experession with f^n, F^n replaced respectively by f, F. Since $\beta_\delta(f)$ is a renormalized solution of (66), this experession is also given (for all $t \ge 0$) by $(\int\int_{\mathbf{R}^{2N}} \beta_\delta(f)^2 \varphi\, dx\, dv)(t)$. In other words, we have

(99) $$\begin{cases} \int\int_{\mathbf{R}^{2N}} \beta_\delta(f^n)^2 \varphi\, dx\, dv \underset{n}{\rightarrow} \int\int_{\mathbf{R}^{2N}} \beta_\delta(f)^2 \varphi\, dx\, dv\ , \\ \qquad \text{uniformly in} \quad t \in [0, T]\ , \end{cases}$$

for all $T \in (0, \infty)$.

In addition, as we saw above, $\beta_\delta(f^n)$ converges to $\beta_\delta(f)$ in $C([0, T]; W^{-s,1}_{loc}(\mathbf{R}^{2N}_{x,v}))$ (for all $s > 1$ and in fact $s = 1$ because of (96)). Therefore, if we consider $L^2_\varphi = L^2$ (Suppφ, φdx), since $(\beta_\delta(f^n))_n$ is bounded in L^2_φ, we deduce that $\beta_\delta(f^n)$ converges uniformly on $[0, T]$ ($\forall\, T \in (0, \infty)$) to $\beta_\delta(f)$ in L^2_φ endowed with the weak topology (represented by a distance on a large ball of L^2_φ). This combined with (99) and the fact that $\beta_\delta(f) \in C([0, \infty); L^2_\varphi)$ implies that $\beta_\delta(f^n)$ converges to $\beta_\delta(f)$ in L^2_φ (strongly) uniformly on $[0, T]$ ($\forall\, T \in (0, \infty)$). And (97) follows.

This concludes the proof of Theorem II.3 since the fact that f is a solution of the (VB) system follows by straightforward limiting arguments that we leave to the reader.

Remark III.1. Another convergence proof is possible using instead the second scheme of proof introduced in Part II [56] in the context of Boltzmann's equation.

We conclude this section by alluding briefly to the proof of Theorem II.1. Indeed, once Theorems II.2 and II.3 are shown, Theorem II.1 follows from a rather standard (and tedious) approximation argument. It is enough to copy the argument presented in [25] for the Boltzmann's equation - with some additional remarks made in Part II [56] - and we skip this more or less trivial adaptation. Let us only mention that one needs to regularize V_0 as well in such a way that (42) holds uniformly. To this end, we consider $\omega \in C_0^\infty(\mathbf{R}^N)$, $\omega \geq 0$, $\int_{\mathbf{R}^N} \omega\, dx = 1$ holds uniformly in ε. Indeed, we apply (42) with $\varphi_\varepsilon = \varphi * \omega_\varepsilon$ and we find that $V_0 * \varphi_\varepsilon$, $\nabla (V_0 * \varphi_\varepsilon)$, $D^2(V_0 * \varphi_\varepsilon)$ are respectively bounded in $L^1_{loc}(\mathbf{R}^N)$, $L^\infty(\mathbf{R}) + L^{\frac{N+2}{3}}(\mathbf{R}^N)$, $L^1_{loc}(\mathbf{R}^N)$. Therefore, the same holds with $V_\varepsilon * \varphi$ since $V_\varepsilon * \varphi = (V_0 * \varphi_\varepsilon) * \omega_\varepsilon$.

IV. Remark on Vlasov-Maxwell-Boltzmann systems

We briefly investigate in this section the Vlasov-Maxwell-Boltzmann system (VMB in short) of (1), ((2)-(4)) and (11)-(14) complemented with the initial conditions (15), (16) that must obey the compatibility condition (17). The formal identities and a priori estimates that we recalled and derived in section II on (VB) systems can be checked for the VB(VMB) system, the main and only modification being the (formal) conservation of the total energy

$$\iint_{\mathbf{R}^6} f|v|^2 dx\, dv + \iint_{\mathbf{R}^3} |E|^2 + |B|^2 dx \quad \text{is independent of} \quad t \geq 0 . \tag{100}$$

On then derives the following a priori estimates

$$\begin{cases} \sup_{t \in [0, T]} \left\{ \iint_{\mathbf{R}^6} f(1 + |v|^2 + \omega(x) + |\log f|)\, dx\, dv + \int_{\mathbf{R}^3} |E|^2 + |B|^2 dx \right\} \\ + \int_0^T dt \int_{\mathbf{R}^3} dx \iint_{\mathbf{R}^6} dv\, dv_* \int_{S^2} d\omega B\, (f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*} \leq C(R, T) \end{cases} \tag{101}$$

if we have

$$\iint_{\mathbf{R}^6} f_0 (1 + |v|^2 + \omega(x) + |\log f_0|)\, dx\, dv + \int_{\mathbf{R}^3} |E_0|^2 + |B_0|^2 dx \leq R . \tag{102}$$

Here, R and T are arbitrary in $(0, \infty)$ and $C(R, T)$ is an (explicit) positive constant which depends only on R and T.

We do not know whether Theorem II.1-II.3 can be adapted or extended to the case of the (VMB) system. In fact, the only information that seems to be missing would be an a priori estimate of E, B in $L^1(0, T; W^{1,1}_{loc}(\mathbf{R}^3))$. If such an estimate were available, then all our analysis would go through and the same results as those stated in section III and proven in section III would hold.

It is also possible to build a solution in a rather weak sense. Indeed, by convenient approximations, one can build a sequence $(f^n, E^n, B^n)_{n\geq 1}$ of approximated solutions (in fact solutions of similar problems with essentially the same structure... - see [25], [27] for more details). And then, one can prove that parts 1)-4) of Theorem II.2 still hold here.

Next, we wish to pass to the limit, as n goes to $+\infty$, in the equations satisfied by $\beta(f^n)$ (we also consider $\beta(f^n-g)$...), E^n, B^n. Of course, we can consider, extracting subsequence if necessary, the weak limit of (f^n, E^n, B^n) denoted by (f, E, B) (respectively in L^1, L^2, L^2) and, clearly, (11)-(14) hold in the sense of distributions. But we also need to describe the weak limits of $\beta(f^n)$, $\beta'(f^n)$ and $\beta'(f^n)$. To this end, it is convenient to introduce the Young's measures $\nu_{x,v,t}$ associated to the sequence f^n in the spirit of L. Tartar's theory of compensated compactness (see L. Tartar [69], [70], R. J. DiPerna [22], [23], R. J. DiPerna and A. Majda [33]...). Indeed, one shows the existence of probability measures $\nu_{x,v,t}$ on $[0, \infty)$ depending measurably on $(x, v, t)\in\mathbf{R}^6_{x,v}\times(0, \infty)$ such that (extracting a subsequence if necessary)

$$\beta(f^n)\underset{n}{\rightarrow}\int\beta(\lambda)\,d\nu_{x,v,t}(\lambda)\qquad\text{weakly in}\quad L^1_{loc}(\mathbf{R}^6_{x,v}\times[0, \infty))\tag{103}$$

for all $\beta\in C([0, \infty); \mathbf{R})$ such that $\beta(t)\,(t\log t)^{-1}\rightarrow 0$ as $t\rightarrow+\infty$. In particular, we have easily

$$\begin{cases} f=\int\lambda\,d\nu_{x,v,t}\ ,\quad f\in L^\infty(0, \infty; L^\infty(\mathbf{R}^6_{x,v}))\ ,\\ \qquad f\in C([0, \infty); L^1(\mathbf{R}^6_{x,v})-w)\end{cases}\tag{104}$$

$$\begin{cases} \sup_{t\in[0,T]}\Big\{\int\int_{\mathbf{R}^6}f(1+|v|^2+\omega(x)+|\log f|)\,dx\,dv\\ +\int\int_{\mathbf{R}^6}dx\,dv\int\lambda|\log\lambda|\,d\nu_{x,v,t}+\int_{\mathbf{R}^3}|E|^2+|B|^2dx\\ +\int_0^T dt\int_{\mathbf{R}^3}dx\int\int_{\mathbf{R}^6}dv\,dv_*\int_{S^2}d\omega B\,(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}\Big\}\leq\infty\ ,\end{cases}\tag{105}$$

for all $T\in(0, \infty)$. In (104), $f\in C([0, \infty); L^1(\mathbf{R}^6_{x,v})-w)$ means that $f(t_n)\underset{n}{\rightarrow}f(t)$ weakly in $L^1(\mathbf{R}^6_{x,v})$ if $t_n(\in[0, \infty))\rightarrow t\in[0, \infty)$. Of course, the initial conditions (15), (16) holds. In addition, we have

$$\nu_{x,v,t}=\delta_{f_0(x,v)}\qquad\text{a.e.}\quad x, v\in\mathbf{R}^6\ .\tag{106}$$

Then, as we said above, parts 1)-4) of Theorem II.2 still hold here and in particular from the proofs made in the preceding section, we can deduce

$$\begin{cases} \frac{\partial}{\partial t}<\beta>+\operatorname{div}_x\{v<\beta>\}+\operatorname{div}_v\{F<\beta>\}\\ \qquad=<\beta'>Q^+(f,f)-<\beta'\lambda>L(f)\qquad\text{in}\quad\mathcal{D}'(\mathbf{R}^6_{x,v}\times(0, \infty))\end{cases}\tag{107}$$

where we denote by $<\varphi>=\int\varphi(\lambda)d\nu_{x,v,t}(\lambda)$ for any $\varphi\in C([0,\infty);\mathbf{R})$ such that $\varphi\{t\log t\}^{-1}$ is bounded on $[2,\infty)$.

In the limiting procedure that yields (107), there is just one technical point to be checked that was not present in the arguments of section III namely the fact that F^n does not converge anymore strongly say in L^1_{loc} (or at least we do not know if it does). One can still pass to the limit using the argument introduced by R. J. DiPerna and P. L. Lions [27] for the Vlasov-Maxwell system: indeed, because of part 1), $\int_{\mathbf{R}^3}\beta(f^n)\psi dv$ converges in $L^p(\mathbf{R}^3_x\times(0,T))$ $(\forall T\in(0,\infty),\ \forall 1\leq p\leq\infty)$ for all $\psi\in C_0^\infty(\mathbf{R}^3_v)$, $\beta\in C([0,\infty);\mathbf{R})$ bounded (or bounded by $C\log(1+t)$...).

The equation (106) clearly holds for all $\beta\in C^1([0,\infty),\mathbf{R})$ such that $\beta'(t)(1+t)$ is bounded on $[0,\infty)$ but with a little more work one can check it holds if $\beta'(t)(1+t)^{1/2}$ is bounded on $[0,\infty)$ and that we have

$$(108)\qquad \begin{cases}\displaystyle\int_0^T dt\int_{\mathbf{R}^3\times K}dx\,dv\Big|Q^+(f,f)\Big(\int\beta'(\lambda)d\nu_{x,v,t}\Big)\\ \displaystyle\qquad\qquad -L(f)\Big(\int\lambda\beta'(\lambda)d\nu_{x,v,t}\Big)\Big|<\infty\end{cases}$$

for all compact sets $K\subset\mathbf{R}^3_v$, $T\in(0,\infty)$.

In addition (46) holds and we have

$$(109)\qquad \frac{\partial\rho}{\partial t}+\operatorname{div}_x(j)=0\quad\text{in}\quad\mathcal{D}'(\mathbf{R}^3_x\times(0,\infty))$$

$$(110)\qquad \int_{\mathbf{R}^6}fv_k dx\,dv\quad\text{is independent of}\quad t\geq 0\quad\text{for}\quad 1\leq k\leq 3$$

$$(111)\qquad \begin{cases}\displaystyle\int\int_{\mathbf{R}^6}f|v|^2dx\,dv+\int_{\mathbf{R}^3}|E|^2+|B|^2dx\\ \displaystyle\qquad\leq\int\int_{\mathbf{R}^6}f_0|v|^2dx\,dv+\int_{\mathbf{R}^3}|E_0|^2+|B_0|^2dx\end{cases}$$

for all $t\geq 0$

This combination of properties (104)-(111) together with (11)-(14) can be used as a definition of a weak solution of the VMB system. And we just saw why there always exists such a weak solution corresponding to the initial conditions (15), (16).

Let us once more emphasize the fact that, if we know that $E,B\in L^1(0,T;W^{1,1}_{loc}(\mathbf{R}^3))$ $(\forall T\in(0,\infty))$, then we can show using the methods of proofs introduced in the preceding section that f is a renormalized solution of the (VMB) system - this relies upon (107) -. Once this is shown, then we can also obtain $<\beta_\delta>=\beta_\delta(f)$ where $\beta_\delta=\frac{1}{\delta}\log(1+\delta t)$. This equality then implies that $\nu_{x,v,t}=\delta_{f(x,v,t)}$ and the a.e. or strong L^1 convergence. Finally, this could then be used to check that f is a solution in a sense analogous to the ones we introduced in section II. Unfortunately, all this argument relies upon the $L_t^1(W_{loc}^{1,1})$ regularity for E, B which does not seem to be available.

The above notion of weak solution can be seen as a precised notion of measure-valued solutions (precised for the collision part of the equation). And the usefulness of such a formulation is not entirely clear. However, it does contain some relevant informations as it can be seen from the following considerations that concern the asymptotics of the (VMB) system when c goes to $+\infty$. We then consider a family (f_0^c, E_0^c, B_0^c) of initial conditions satisfying (17) and the bound (102) uniformly in c. Then, we introduce (f^c, ν^c, E^c, B^c) solutions of the (VMB) system corresponding to the initial conditions (f_0^c, E_0^c, B_0^c) satisfying (105) uniformly in c. Therefore, we may assume without loss of generality, extracting subsequences if necessary, that f^c, E^c, B^c converge weakly, respectively in $L^1(\mathbf{R}^6_{x,v}\times(0,T))$, $L^2(\mathbf{R}^3_x\times(0,T))$, $L^2(\mathbf{R}^3_x\times(0,T))$ $(\forall T\in(0,\infty))$ to f, E, B and that ν^c converges weakly in the sense of measures to some ν which is a probability measure on $[0,\infty)$ parametrized (measurably) by $(x,v,t)\in\mathbf{R}^6\times[0,\infty)$. Clearly, (104) - (106) hold by the same arguments than those introduced in section III (see also the above comments) with $F=E$ and (11) holds where f_0 is the weak limit (in $L^1(\mathbf{R}^6)$) of f_0^c. In addition, parts 1) -4) of Theorem II.2 also hold here (with f^n replaced by f^c) and we have

$$\int_{\mathbf{R}^3}\int\beta(\lambda)\psi(v)\,d\nu^c_{x,v,t}(t)\to\int_{\mathbf{R}^3}\int\beta(\lambda)\psi(v)\,d\nu_{x,v,t}(\lambda)\quad\text{as } c\to+\infty\ ,\tag{112}$$

in $L^p(0,T;L^1(\mathbf{R}^3_x))$ $(\forall\, 1\le p<\infty,\ \forall\, T\in(0,\infty))$, for all $\beta\in C^1([0,\infty);\mathbf{R})$ such that $\beta(0)=0$ and $\beta'(t)$ is bounded on $[0,\infty)$ and for all $\psi\in L^\infty(\mathbf{R}^3_v)$ such that $\psi(v)\,(1+|v|^2)^{-1}\to0$ as $|v|\to+\infty$ ($\operatorname{infess}_{|v|\ge R}\{|\psi(v)|(1+|v|^2)^{-1}\}\to0$ as $R\to+\infty$).

We next claim that $f\in C([0,\infty);L^1(\mathbf{R}^6_{x,v}))$ is a renormalized solution of the (VPB) system or in other words that $F\equiv E\equiv-\nabla V$ where V solves

$$-\Delta V=\rho\Big(=\int_{\mathbf{R}^3}f\,dv\Big)\ ,\quad V\in L^\infty(0,\infty;L^{3,\infty}(\mathbf{R}^3_x))\ .\tag{113}$$

In order to prove this claim, in view of the arguments presented in section III, we only have to understand why $E\equiv-\nabla V$ with V solution of (113) (i.e. $V=\frac{1}{4\pi}\frac{1}{|x|}*\rho$). But, if we pass to the limit in the sense of distributions in (12), (13) using (112), we find

$$\operatorname{curl}B=\operatorname{div}B=\operatorname{curl}E=0\ ,\quad \operatorname{div}E=\rho\qquad\text{in }\mathcal{D}'\ .\tag{114}$$

Hence, $B\equiv0$ (recall that $B\in L^\infty(0,\infty;L^2(\mathbf{R}^3_x))$) and (113) holds.

Furthermore, if we assume that f_0^c converges strongly in $L^1(\mathbf{R}^6)$ to f_0, then we deduce from the arguments of section III that $\nu_{x,v,t} = \delta_{f(x,v,t)}$ and that $\int \beta(\lambda)\, d\nu^c_{x,v,t}$ converges in $L^1(\mathbf{R}^6_{x,v})$ uniformly in $t \in [0, T]$ $(\forall T \in (0, \infty))$ to $\beta(f)$ for all $\beta \in C^1([0, \infty))$ such that $\beta(0) = 0$, β' is bounded on $[0, \infty)$. In particular, f^c $(= \int \lambda d\nu^c_{x,v,t})$ converges to f in $L^1(\mathbf{R}^6_{x,v})$ uniformly in $t \in [0, T]$ for all $T \in (0, \infty)$.

V. On Boltzmann-Dirac model

We consider in this section the Boltzmann-Dirac model (BD in short) namely equation (1) with $F \equiv 0$ and Q given by (24) where the collision kernel B still satisfies (5) - (7). Of course, we complement this system with an initial condition (15) where $f_0 \geq 0$ satisfies

$$(115) \qquad 0 \leq f_0 \leq \varepsilon^{-1} \quad \text{a.e. on } \mathbf{R}^{2N}, \quad \iint_{\mathbf{R}^{2N}} f_0 (1 + |v|^2 + \omega(x))\, dx\, dv < \infty .$$

Then, at least formally, we expect to find a solution f which satisfies

$$(116) \qquad 0 \leq f \leq \varepsilon^{-1} \quad \text{a.e. on } \mathbf{R}^{2N}_{x,v} \times [0, \infty) .$$

This fact follows easily from simple differential equations considerations (maximum principle, notice indeed that $Q(f, f) \leq 0$ at a point (x, v, t) where f is equal to ε while $Q(f, f) \geq 0$ at a point where f is equal to 0).

This bound explains why, in some sense, the (BD) model is somewhat better behaved than the Boltzmann model. And in fact, it was shown in J. M. Dolbeault [34] that the existence, uniqueness and regularity of solutions is available when $B \in L^1(\mathbf{R}^N \times S^{N-1})$. However, if we drop this requirement, the situation is less clear and the non-quadratic nature of the collision operator creates additional difficulties for weak passages to the limit. In particular, the method introduced in [25] (or in section II for the proof of Theorem II.2) for the Boltzmann model does not seem to carry over the (BD) model.

We resolve this problem in this section where we prove a general existence theorem based upon a Fourier analysis of various parts of the collision operator. In some sense, this analysis relies upon the analysis performed in Part I [55] via Fourier integral operators.

Let us first state precisely our main results. We begin by recalling that, but for the entropy, the same conservation laws that for Boltzmann model are available here namely (29), (30), (31) (with $V_0 \equiv 0$) and in particular (32) - (34) still hold here.

Furthermore, we have (at least formally)

$$(117) \qquad \frac{d}{dt} \iint_{\mathbf{R}^{2N}} f \omega\, dx\, dv = \iint_{\mathbf{R}^{2N}} f \{v \cdot \nabla \omega(x)\}\, dx\, dv$$

and

(118) $$\iint_{\mathbf{R}^{2N}} f|x-vt|^2 dx\, dv \quad \text{is independent of} \quad t \geq 0 \ .$$

Finally, the entropy identity (38) is now replaced by

(119) $$\begin{cases} \frac{d}{dt}\iint_{\mathbf{R}^{2N}} f \log f + \frac{1}{\varepsilon}(1-\varepsilon f)\log(1-\varepsilon f)\, dx\, dv \\ +\frac{1}{4}\int_{\mathbf{R}^N} dx \int_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} d\omega B (1-\varepsilon f)(1-\varepsilon f_*)(1-\varepsilon f')(1-\varepsilon f'_*) \\ \cdot \left\{ \frac{f'}{1-\varepsilon f'} \frac{f'_*}{1-\varepsilon f'_*} - \frac{f}{1-\varepsilon f} \frac{f_*}{1-\varepsilon f_*} \right\} \\ \cdot \log \left[\frac{f'}{1-\varepsilon f'} \frac{f'_*}{1-\varepsilon f'_*} \frac{1-\varepsilon f}{f} \frac{1-\varepsilon f_*}{f_*} \right] = 0 \qquad \text{for} \quad t \geq 0 \ . \end{cases}$$

Notice that the second term is nonnegative.

It is worth emphasizing the fact - justified in [34] when $B \in L^1$ - that we recover the Boltzmann model if we let ε go to 0_+ in which case (119) reduces to (38) (using the conservation of mass (32)). Observing that $\frac{1}{\varepsilon}(1-\varepsilon t)\log(1-\varepsilon t) \geq -t$ on $[0, \infty)$, we deduce easily - as in section II - the following a priori bounds

(120) $$\sup_{t\in[0,T]} \left\{ \iint_{\mathbf{R}^{2N}} f \{1+|v|^2+\omega(x)\}\, dx\, dv \right\} < \infty \ , \quad \text{for all } T \in (0, \infty)$$

(121) $$\begin{cases} \int_0^T dt \int_{\mathbf{R}^N} dx \int_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} d\omega B (1-\varepsilon f)(1-\varepsilon f_*) \cdot \\ (1-\varepsilon f')(1-\varepsilon f'_*) \{F'F'_* - FF_*\} \log \frac{F'F'_*}{FF_*} < \infty, \text{ for all } T \in (0, \infty) \ , \end{cases}$$

where we set $F = \frac{f}{1-\varepsilon f}$.

In addition, if we take $\omega(x) = |x|^2$ in (115) then (120), (121) hold with $T = +\infty$ replacing $\omega(x)$ by $|x-vt|^2$ in (120). Finally, the bounds in (120), (121) depend only on the bound in (115).

We then define solutions of the (BD) model as follows: $f \in C([0, \infty); L^p(\mathbf{R}^{2N}_{x,v}))$ for all $1 \leq p < \infty$, satisfies the (BD) equation (in the sense of distributions for instance), (116), (120), (121). Notice that $Bff_*(1-\varepsilon f')(1-\varepsilon f'_*) \in L^\infty(0, T; L^1(\mathbf{R}^N_x \times K \times \mathbf{R}^N_{v*} \times S^{N-1}_\omega))$ for all $T \in (0, \infty)$, K compact set $\subset \mathbf{R}^N_v$ and thus $Bf'f'_*(1-\varepsilon f)(1-\varepsilon f_*) \subset L^1(\mathbf{R}^N_x \times K \times \mathbf{R}^N_{v*} \times S^{n-1}_\omega \times (0, T))$ (for all $T \in (0, \infty)$) because of (121). Thus $Q(f) \in L^1(\mathbf{R}^N_x \times K \times (0, T))$ for all compact sets $K \subset \mathbf{R}^N_v$, $T \in (0, \infty)$, and the equation makes sense. We could complement this formulation with various conservation laws (like (29), (32), (33), (117)) or even more complicated identities involving multipliers like the ones introduced in section II but we shall not do here in order to simplify the presentation. Also, we could request energy and entropy inequalities

(122) $$\iint_{\mathbf{R}^{2N}} f(x, v, t)\,|v|^2 dx\,dv \leq \iint_{\mathbf{R}^{2N}} f|v|^2 dx\,dv\ , \quad \text{for all} \quad t \geq 0\ ,$$

(123) $$\begin{cases} \displaystyle\iint_{\mathbf{R}^{2N}} f\log f + \frac{1}{\varepsilon}(1-\varepsilon f)\log(1-\varepsilon f)\,dx\,dv \\ \displaystyle + \int_0^t ds \int_{\mathbf{R}^N} dx \int_{\mathbf{R}^{2N}} dv\,dv_* \int_{S^{N-1}} d\omega B \cdot \\ \displaystyle \cdot (1-\varepsilon f)(1-\varepsilon f_*)(1-\varepsilon f')(1-\varepsilon f'_*)\,\{F'F'_* - FF_*\}\log\frac{F'F'_*}{FF_*} \leq \\ \displaystyle \iint_{\mathbf{R}^{2N}} f_0\log f_0 + \frac{1}{\varepsilon}(1-\varepsilon f_0)\log(1-\varepsilon f_0)\,dx\,dv\ , \quad \text{for all} \quad t \geq 0\ . \end{cases}$$

Our main result is the

Theorem V.1. *Let* f_0 (115), *then there exists a solution* f *of the* (*BD*) *model satisfying the initial conditions* (15) *and the a priori bounds* (120) - (121).

Theorem V.1 is a straightforward consequence of stability results we describe now: we consider a sequence of initial conditions $(f_0^n)_{n\geq 1}$ satisfying (115) uniformly in n. And we consider a sequence of solutions of the (BD) model denoted by $(f^n)_{n\geq 1}$ that corresponds to the initial condition f_0^n. In addition, we assume that f^n satisfies (120), (121) uniformly in n. The existence of such a sequence follows in fact from Theorem V.1. Extracting subsequences if necessary, we may assume without loss of generality that f_0^n, f^n converge weakly in $L^1(\mathbf{R}^{2N}_{x,v})$ and $L^\infty(\mathbf{R}^{2N}_{x,v}) - *$, in $L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$ $(\forall\, T \in (0, \infty))$ and $L^\infty(\mathbf{R}^{2N}_{x,v} \times (0, \infty)) - *$ respectively to f_0, f. In the result that follows, we denote by

(124) $$Q^+(f) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B(1-\varepsilon f)(1-\varepsilon f_*) f' f'_*$$

(125) $$Q^-(f) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B(1-\varepsilon f')(1-\varepsilon f'_*) f f_*\ .$$

Our main stability result is the following

Theorem V.2. *The weak limit* f *is a solution of the* (BD) *model satisfying* (15). *Furthermore, we have*

$$\int_{\mathbf{R}^N} \beta(f^n)\,\psi dv \quad \text{is relatively compact in} \quad L^1(\mathbf{R}^N_x \times (0, T)) \quad (\forall\, T \in (0, \infty))$$

for all $\beta \in C([0, \infty), \mathbf{R})$ *such that* $\beta(t)t^{-1}$ *is bounded near* 0 *and for all* $\psi \in L^\infty_{loc}(\mathbf{R}^N)$ *such that* $\psi(v)(1+|v|^2)^{-1} \longrightarrow 0$ *as* $|v| \to \infty$,

(127) $$L(f^n) \underset{n}{\rightarrow} L(f) \quad \text{in} \quad L^p(0, T; L^1(\mathbf{R}^N_x \times K)),\ \forall\, 1 \leq p < \infty,$$

$$\text{(128)} \qquad \begin{cases} Q^+(f^n),\ Q^-(f^n) \quad \textit{converge weakly to} \quad Q^+(f),\ Q^-(f) \ \textit{respectively}, \\ \textit{in } L^1(\mathbf{R}_x^N \times K \times (0,T)) \ \textit{for all } T \in (0,\infty),\ \textit{compact sets } K \subset \mathbf{R}_v^N\ . \end{cases}$$

Remark V.1. i) We do not know if solutions are unique, nor if they are more regular if f_0 is more regular.

ii) Theorem V.1 is still valid if we let the collision kernel B depend on n provided B^n satisfies (5), (6) for all $n \geq 1$, (7) uniformly in n and B^n converges to B in $L^1(K \times S^{N-1})$ for all compact sets $K \in \mathbf{R}^N$.

iii) We could also consider problems set in a periodic box. The results would be identical and we could consider the behavior of solutions when t goes to $+\infty$.

iv) It is possible to study the limit when ε goes to 0_+ and recover solutions of Boltzmann's equation. But we shall not do so here.

In view of the preceding remark (ii), the existence result (Theorem V.1) follows from Theorem V.2: indeed, it is enough to truncate B, apply the results of J. M. Dolbeault [34] to obtain solutions for the resulting equations and pass to the limit using Theorem V.2.

Before we really begin the proof of Theorem V.2, we wish to make a few preliminary remarks. First of all, (126) - (128) are easy to show (with similar arguments, and in fact simpler, than in [25] or in section III). Let us only point out that (128) follows from the fact that $0 \leq Q^-(f^n) \leq \frac{1}{\varepsilon} L(f^n)$ while the weak compactness in L^1 of Q^+ is deduced from the weak compactness in L^1 of Q^- and from the entropy dissipation bound (121). Then, (126), (127) follow from velocity averaging results and the bounds (120).

Then, without loss of generality, we may assume that $Q^+(f^n)$, $Q^-(f^n)$ converge weakly in $L^1(\mathbf{R}_x^N \times K \times (0,T))$ (for all compact sets $K \subset \mathbf{R}_v^N$, $T \in (0,\infty)$) to, respectively, Q^+, $Q^- \geq 0$. And we have of course: $f \in C([0,\infty); L^p(\mathbf{R}_{x,v}^{2N}))$ satisfies (120), (15) and

$$\text{(129)} \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q^+ - Q^- \quad \text{in} \quad \mathscr{D}'(\mathbf{R}_{x,v}^{2N} \times (0,\infty))\ .$$

Furthermore, adapting the argument used in section III, we have

$$\text{(130)} \qquad Q^+ \leq KQ^- + \frac{1}{\log K} e_0\ , \quad Q^- \leq KQ^+ + \frac{1}{\log K} e_0 \qquad \text{a.e.}$$

for some $e_0 \geq 0$, $e_0 \in L^1(\mathbf{R}_{x,v}^{2N} \times (0,T))$ $(\forall\, T \in (0,\infty))$.

In order to complete the proof of Theorem V.2, these remains to show that $Q^+ - Q^- = Q(f)$ or in fact that $Q^+ = Q^+(f)$, $Q^- = Q^-(f)$. The proof of these two claims is divided into several steps. The first one consists in approximating B conveniently. We truncate B and consider $B_\delta(z,\omega)$ $(0 < \delta \leq 1)$ a nonnegative function of $|z|$ and $|(z, \omega)|$ only such that

(131) $B_\delta \equiv 0$ for $|z|$ small, $|z|$ large, $|z \cdot \omega|$ small, $|z| - |(z, \omega)|$ small

(132) $0 \leq B_\delta(z, \omega) \leq B(z, \omega)$

(133) $B_\delta(z, \mu) \uparrow B(z, \omega)$ as $\delta \downarrow 0_+$, a.e. $(z, \omega) \in \mathbf{R}^N \times S^{N-1}$.

We denote by Q_δ, Q_δ^+, Q_δ^- the associated collision operators defined by replacing B by B_δ in Q, Q^+, Q^- respectively.

We claim that we have

$$(134) \quad \begin{cases} \sup_{n \geq 1} \{\|Q^+(f^n) - Q_\delta^+(f^n)\|_{L^1(C)} + \|Q^-(f^n) - Q_\delta^-(f^n)\|_{L^1(C)}\} \\ \qquad \to 0 \quad \text{as} \quad \delta \to 0_+ \end{cases}$$

where $C = \mathbf{R}_x^N \times K \times (0, T)$, for all compact sets $K \subset \mathbf{R}_v^N$, $T \in (0, \infty)$. Obviously $Q^+ - Q_\delta^+$, $Q^- - Q_\delta^-$ are also collision operator that corresponds to the collision kernel $(B - B_\delta)$ which satisfies: $0 \leq B - B_\delta \leq B$. Therefore, it is enough to prove (134) for $Q^- - Q_\delta^-$ since the other assertion then follows using (121). Next, we observe that we have

$$0 \leq Q^-(f^n) - Q_\delta^-(f^n) \leq \frac{1}{\varepsilon}(A - A_\delta) \underset{v}{*} f^n \qquad \text{a.e.}$$

where $A_\delta(v) = \int_{S^{N-1}} B_\delta(v, \omega))d\omega$. Therefore

$$\begin{cases} \|Q^-(f^n) - Q_\delta^-(f^n)\|_{L^1(c)} \leq \\ \frac{1}{\varepsilon}\int\int_{\mathbf{R}^N \times (0, T)} dx\, dt\, f^n(x, v_*, t) \int_K (A - A_\delta)(v - v_*) dv \ . \end{cases}$$

Then, because of (133), $\int_K (A - A_\delta)(v - v_*) dv$ converges to 0 uniformly in v_* and is bounded as δ goes to 0_+. And since, by assumption on B.

$$\begin{cases} 0 \leq \int_K (A - A_\delta)(v - v_*) dv \\ \ \leq \int_K A(v - v_*) dv = o(|v_*|^2) \qquad \text{as} \quad |v_*|^2 \to \infty \ , \end{cases}$$

we conclude easily that (134) holds.

We next wish to show that $Q_\delta(f^n)$ converges weakly in $L^\infty(\mathbf{R}_{x,v}^{2N} \times (0, \infty)) - *$ or in $L^p(\mathbf{R}_{x,v}^{2N} \times (0, T))$ for all $T \in (0, \infty)$, $1 \leq p < \infty$, to $Q_\delta(f)$. Notice that $0 \leq Q_\delta^\pm(f^n) \leq \frac{1}{\varepsilon^2} \|B_\delta\|_{L^1(\mathbf{R}^N \times S^{N-1})}$, (121) holds and $Q_\delta^-(f^n) \leq \frac{1}{\varepsilon}\|B_\delta\|_{L^1(\mathbf{R}^N \times S^{N-1})} f^n$, and thus the weak convergences are obvious. However, we have to identify the weak limits and this is a priori delicate since Q_δ is highly nonlinear. Assuming that this claim has been proven, we see that f is indeed a solution of the BD model or, in other words, that $Q^+ - Q^- = Q(f)$. For, because of (134), $Q_\delta(f)$ converges to $Q^+ - Q^-$ in $L^1(\mathbf{R}^N_x \times K \times (0, T))$ as δ goes to 0_+ (for all compact sets $K \subset \mathbf{R}^N_v$, $T \in (0, \infty)$). On the other hand, adapting the argument made above and using in particular (130), we check that $Q_\delta(f)$ converges to $Q(f)$ in $L^1(\mathbf{R}^N_x \times K \times (0, T))$ ($\forall K$ compact $\subset \mathbf{R}^N_v$, $\forall T \in (0, \infty)$).

In order to prove that the weak limit of $Q_\delta(f^n)$ that we denote by Q_δ is nothing but $Q_\delta(f)$, we decompose $Q_\delta(\cdot)$ in the following way: we write for all functions g

(135) $$Q_\delta(g) = Q^1_\delta(g) - gR_\delta(g)$$

where Q^1_δ, $R_\delta \geq 0$ are given by

(136) $$Q^1_\delta(g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B_\delta g' g'_* (1 - \varepsilon g_*)$$

(137) $$R_\delta(g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B_\delta \{g_* (1 - \varepsilon g')(1 - \varepsilon g'_*) + \varepsilon g' g'_* (1 - \varepsilon g_*)\} .$$

Of course, we may rewrite Q^1_δ and R_δ as follows:

(138) $$Q^1_\delta(g) = Q^+_\varepsilon(g, g) - \varepsilon Q^2_\delta(g) \ , \quad Q^2_\delta(g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B_\delta g' g'_* g_*$$

where $Q^+_\delta(g, g)$ is the usual (quadratic) "gain" of the Boltzmann's collision operator, and

(139) $$R_\delta(g) = L_\delta(g) - \varepsilon Q^3_\delta(g, g) - \varepsilon Q^4_\delta(g, g) + \varepsilon Q^+_\delta(g, g)$$

where

(140) $$Q^3_\delta(g, g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B_\delta g_* g'$$

(141) $$Q^4_\delta(g, g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B_\delta g'_* g'$$

Observe also that $Q^+_\delta(f^n, f^n)$, $Q^3_\delta(f^n, f^n)$, $Q^4_\delta(f^n, f^n)$, $Q^2_\delta(f^n)$ are clearly bounded in $L^\infty(\mathbf{R}^{2N}_{x,v} \times (0, \infty))$ (by $C_\varepsilon \|B_\delta\|_{L^1}$) and also weakly compact in $L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$ ($\forall T \in (0, \infty)$). We skip these easy arguments somewhat similar (and simpler!) to many ones we did before.

The above claim about Q_δ will be a consequence of the following fact: for each $\delta > 0$, we claim that we have

(142) $$Q^1_\delta(f^n) \underset{n}{\rightarrow} Q^1_\delta(f) \ , \quad R_\delta(f^n) \underset{n}{\rightarrow} R_\delta(f) \quad \text{in } L^p(\mathbf{R}^{2N}_{x,v} \times (0, T))$$

and

(143) $$\int Q^2_\delta(f^n) \psi dv \underset{n}{\rightarrow} \int Q^2_\delta(f) \psi dv \qquad \text{in} \quad L^p(\mathbf{R}^N_x \times (0, T))$$

for all $1 \leq p < \infty$, $T \in (0, \infty)$, $\psi \in C_0^\infty(\mathbf{R}_v^n)$.

Therefore, we only have to prove (142) and (143) which, in turn, is a consequence of the following convergences

$$(144) \qquad \begin{cases} Q_\delta^+(f^n, f^n) \underset{n}{\rightarrow} Q_\delta^+(f, f) \quad , \quad Q_\delta^3(f^n, f^n) \underset{n}{\rightarrow} Q_\delta^3(f, f) \\ Q_\delta^4(f^n, f^n) \underset{n}{\rightarrow} Q_\delta^4(f, f) \end{cases}$$

$$(145) \qquad \int_{\mathbf{R}^N} Q_\delta^2(f^n)\, \psi\, dv \underset{n}{\rightarrow} \int_{\mathbf{R}^N} Q_\delta^2(f)\, \psi\, dv$$

in L^p, for all $1 \leq p < \infty$, $T \in (0, \infty)$.

In addition, by a simple density argument, we see that it is enough to show (144) and (145) in the case when $B_\delta \in C_0^\infty(\mathbf{R}^N \times S^{N-1})$ (or even more precisely is a C^∞ function of $|v|$, (v, ω) that vanishes for $|v|$ small, $|v|$ large, $|(v, \omega)|$ small, $|v| - |(v, \omega)|$ small). From now on, we make this assumption and we omit the subscript δ in all that follows in order to simplify notations.

In order to prove (144) and (145), we want to apply the results of Part I [55]. We begin with the three quadratic terms Q^+, Q^3, Q^4. The strategy and the results are the same for each of these terms: we are going to show that velocity averages of these terms are compact (in L^1 or in L^2) in (x, t) and then we shall prove that they belong to $L^2(\mathbf{R}_x^N \times (0, T);\, H_{loc}^{\frac{N-1}{2}}(\mathbf{R}_v^N))$ $(\forall\, T \in (0, \infty))$. This is enough to conclude that (144) holds by the arguments of section II in [55] that $Q^+(f^n, f^n)$, $Q^3(f^n, f^n)$, $Q^4(f^n, f^n)$ are bounded in $L^\infty(\mathbf{R}_{x,v}^{2N} \times (0, \infty))$ and weakly compact in $L^1(\mathbf{R}_{x,v}^{2N} \times (0, T))$ for all $T \in (0, \infty)$, so we only need the compactness in L_{loc}^1 in order to prove (145).

Then, we need first to show that for all $\psi \in C_0^\infty(\mathbf{R}_v^N)$

$$(146) \qquad \begin{cases} \int_{\mathbf{R}^N} Q^+(f^n, f^n)\, \psi\, dv,\ \int_{\mathbf{R}^N} Q^3(f^n, f^n)\, \psi\, dv,\ \int_{\mathbf{R}^N} Q^4(f^n, f^n)\, \psi\, dv \\ \text{converge respectively to} \\ \int_{\mathbf{R}^N} Q^+(f, f)\, \psi\, dv,\ \int_{\mathbf{R}^N} Q^3(f, f)\, \psi\, dv,\ \int_{\mathbf{R}^N} Q^4(f, f)\, \psi\, dv \end{cases}$$

in $L_{loc}^1(\mathbf{R}^N \times (0, \infty))$ (say!). This will be in fact a straightforward consequence of (126) choosing $\beta(t) \equiv t$ (and observing that $\int_{\mathbf{R}^N} f^n \psi dv$ is relatively compact in $L^2(\mathbf{R}_x^N \times (0, T))$ $(\forall\, T \in (0, \infty))$ and thus converges in $L^2(\mathbf{R}_x^N \times (0, T))$ $(\forall\, T \in (0, \infty))$ to $\int_{\mathbf{R}^N} f \psi dv$ for all $\varphi \in C_0^\infty(\mathbf{R}_v^N)$. Indeed, we only have to show that the three integrals in v above (velocity averages) can be written as

$$\int_{\mathbf{R}^{2N}} f^n(x, v, t) f^n(x, v, t)\, a(v, w)\, dv\, dw$$

for some $a \in C_0^\infty(\mathbf{R}^{2N})$. Then, (146) follows easily.

Such a representation is easy for Q^+ since we have changing variables $(v' \to v,\ v'_* \to v_*)$

$$\int_{\mathbf{R}^N} Q^+(f^n, f^n)\,\phi\,dv = \int\int_{\mathbf{R}^{2N}} f^n(x, v, t) f^n(x, v_*, t) \cdot \left\{\int_{S^{N-1}} B(v - v_*, \omega)\,\phi(v')\,d\omega\right\} dv\,dv_* .$$

The representation for Q^4 is obtained in a similar way (in fact simpler) to the one for Q^3 so we only detail it for Q^3. We first remark that, by the same change of variables, we find

$$\int_{\mathbf{R}^N} Q^3(f^n, f^n)\,\phi\,dv = \int\int_{\mathbf{R}^{2N}\times S^{N-1}} f^n(x, v, t) f^n(x, v'_*, t)\,\phi(v') \cdot B(v - v_*, \omega)\,dv\,dv_*\,d\omega .$$

Then, the idea is that, roughly speaking, v'_* describes $\mathbf{R}^N$ when (v_*, ω) describes $\mathbf{R}^N \times S^{N-1}$ with in fact $(N-1)$ free paramenters for each fixed v. More precisely, on can check that the mapping from $S = \{(v, \omega) \in \mathbf{R}^N \times S^{N-1} / v_* \cdot \omega = 0,\ v \cdot \omega > 0\}$ into $\mathbf{R}^N - \{v\}$ defined by $v'_* = v_* + (v, \omega)\omega$ is smooth and $1-1$. The representation we look for is thus achieved by a simple change of variables: notice indeed that $B(v - v_*, \omega)$ vanishes near $\{(v_*, \omega) = 0,\ (v, \omega)$ small$\}$ and that $\{v\}$ is the image of $\{(v_*, \omega) \in S / |v - v_*| = |(v - v_*, \omega)|\}$ in a neighborhood of which B vanishes. Therefore, the change of variables is smooth over the support of B and we conclude.

In order to complete the proof of (144), we still have to show that $Q^+(f^n, f^n)$, $Q^3(f^n, f^n)$, $Q^4(f^n, f^n)$ are bounded in $L^2(\mathbf{R}_x^N \times (0, T); H_{loc}^{\frac{N-1}{2}})$. For Q^+, this is a simple consequence of the results of Part I [55] since we have for all x, t

$$\|Q^+(f, f)\|_{H^{\frac{N-1}{2}}(\mathbf{R}_v^N)} \leq C \|f\|_{L^2(\mathbf{R}_v^N)} \|f\|_{L^1(\mathbf{R}_v^N)} \tag{147}$$

for some $C \geq 0$ independent of f. Of course, we deduce from (147) for each $R \in (0, \infty)$

$$\|Q^+(f, f)\|_{H^{\frac{N-1}{2}}(B_R)} \leq C \|f\|_{L^2(B_{R+R_0})} \|f\|_{L^1(B_{R+R_0})} \tag{148}$$

for some R_0 (that depends only on the size of the support of B), where $B_M = \{v \in \mathbf{R}^N / |v| < M\}$. And our claim on $Q^+(f^n, f^n)$ is shown since $(\int_{B_{R_0+R}} |f|^2 dv)^{1/2} \cdot (\int_{B_{R_0+R}} |f| dv) \in L^1(\mathbf{R}_x^N \times (0, T)) \cap L^\infty(\mathbf{R}_x^N \times (0, \infty))$ $(\forall\, T \in (0, \infty))$. The above claims on Q^3, Q^4 are proven in the same way in view of the result that follows which shows that the results shown in Part I [55] on Q^+ also hold for Q^3 and Q^4. Let us recall that in the result which follows, $B = B(z, \omega)$ is a smooth function over $\mathbf{R}^N \times S^{N-1}$ that vanishes near $z = 0$, $z \cdot \omega = 0$, $|z| - |(z, \omega)| = 0$,

$|z|$ large and which depends only on $|z|$ and $|(z, \omega)|$. We denote by

(149) $$Q^3(f, g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B(v - v_*, \omega) f(v_*) g(v')$$

(150) $$Q^4(f, g) = \int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B(v - v_*, \omega) f(v_*) g(v'_*)$$

for all $f, g \in C_0^\infty(\mathbf{R}_v^N)$. We then have the

Proposition V.1. *Q^3 and Q^4 are bounded bilinear maps from $L^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ into $H^{\frac{N-1}{2}}(\mathbf{R}^N)$.*

Remark V.2. Exactly as in Part I [55], more general results are possible with L^p spaces. In particular, we can replace $L^1(\mathbf{R}^N)$ by the space of bounded measures on $\mathbf{R}^N$. Also, Q^3 and Q^4 are bounded from $L^1(\mathbf{R}^N) \times H^s(\mathbf{R}^N)$ into $H^{s+\frac{N-1}{2}}(\mathbf{R}^N)$ for all $s \in \mathbf{R}$.

Proof of Proposition V.1. We simply observe that by changing variables $[(v, v_*) \to (v', v'_*)]$ we find for all $\varphi \in C_0^\infty(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} Q^3(f, g)\varphi \, dv = \int_{\mathbf{R}^N} g(v) dv \left\{ \int\int_{\mathbf{R}^N \times S^{N-1}} B(v - v_*, \omega) \varphi(v') f(v'_*) dv_* d\omega \right\}$$
$$= \int_{\mathbf{R}^N} g(v) Q^+(\varphi, f) dv$$

and similarly changing variables $[(v, v_*) \to (v'_*, v')]$

$$\int_{\mathbf{R}^N} Q^4(f, g)\varphi dv = \int_{\mathbf{R}^N} g(v) dv \left\{ \int\int_{\mathbf{R}^N \times S^{N-1}} B(v - v_*, \omega) f(v') \varphi(v'_*) dv_* d\omega \right\}$$
$$= \int_{\mathbf{R}^N} g(v) Q^+(f, \varphi) dv \ .$$

And we conclude easily using the results of Part I [55] since Q^+ is bounded from $H^{-\frac{N-1}{2}}(\mathbf{R}^N) \times L^1(\mathbf{R}^N)$ and $L^1(\mathbf{R}^N) \times H^{-\frac{N-1}{2}}(\mathbf{R}^N)$ into $L^2(\mathbf{R}^N)$.

There remains to prove (145). First of all, we consider $\psi \in C_0^\infty(\mathbf{R}_v^N)$ and rewrite exchanging v and v_*

(151) $$\begin{cases} \int_{\mathbf{R}^N} Q^3(f^n)\psi \, dv = \int_{\mathbf{R}^N} f^n(x, v, t) \\ \cdot \left\{ \int\int_{\mathbf{R}^N \times S^{N-1}} B(v - v_*, \omega) f^n(x, v', t) f^n(x, v'_*, t) \psi(v_*) dv_* d\omega \right\} dv \ . \end{cases}$$

The quantity between brackets has exactly the same structure that $Q^+(f^n, f^n)$ and the same proof as above yields

$$\begin{cases} \iint_{\mathbf{R}^N\times S^{N-1}} Bf^{n'}{}_* f^{n'}_* \psi_* dv_* d\omega \xrightarrow{n} \iint_{\mathbf{R}^N\times S^{N-1}} Bf' f'_* \psi_* dv\, d\omega \\ \quad \text{in} \quad L^p(\mathbf{R}^N_x \times (0, T)) \end{cases}$$

for all $T\in(0,\infty)$, $1\le p<\infty$. We then use (126) to deduce from (152) the convergence (145). And we conclude.

Remark V.3. It is natural to ask whether (145) can be improved and in particular whether $Q^2(f^n)$ can be (automatically) compact in $L^1_{x,v,t}$ (locally). In fact, this is not the case. Indeed, one can, equivalently, consider for complex-valued functions f^n, g^n, h^n

$$Q^2(f^n, g^n, h^n) = \iint_{\mathbf{R}^N\times S^{N-1}} dv_* d\omega\, B f^n_* g^{n'} h^{n'}_* \; .$$

We then choose $f^n = e^{-inv_1}\varphi$, $g^n = e^{inv_1}\varphi$, $h^n = e^{inv_1}\varphi$ where $n\ge 0$, $\varphi\in C_0^\infty(\mathbf{R}^N_v)$ is such that $Q^2(\varphi,\varphi,\varphi)\not\equiv 0$. Then, we observe that we have

$$Q^2(f^n, g^n, h^n) = e^{inv_1} Q^2(\varphi,\varphi,\varphi)$$

and this sequence is not relatively compact in $L^1_{loc}(\mathbf{R}^{2N}_{x,v}\times(0,\infty))$.

Appendix 1. $L\log L$ integrability of averages

We show in this appendix the following

Lemma. *Let $f\ge 0$ satisfy*

$$\text{(A.1)} \qquad A = \iint_{\mathbf{R}^N\times\mathbf{R}^k} f(x,y)\,(1+|v|^2+\omega(x)+|\log f|)\,dx\,dy<\infty$$

where ω satisfies (19). *Let $\rho(x) = \int_{\mathbf{R}^k} f(x,y)\,dy$. Then, we have*

$$\text{(A.2)} \qquad \int_{\mathbf{R}^N} \rho(x)\,|\log\rho(x)|\,dx \le C_0 A$$

for some $C_0>0$ independent of f.

Proof. In view of the Appendix 1 of Part I [55], it is enough to show

$$\text{(A.3)} \qquad \int_{\mathbf{R}^N} \rho\log\rho\, dx \le (1+\pi)A$$

since (A.1) implies obviously: $\rho\omega\in L^1(\mathbf{R}^N)$.

In order to prove (A.3), we use a classical trick (in Stastical Physics) and we recall first the following convexity inequality valid for all $a, b\in[0,\infty)$

$$\text{(A.4)} \qquad a\log a \ge a\log b + a - b \; .$$

We then apply (A.4) with $a=f(x,y)$, $b=\rho(x)e^{-\pi|y|^2}$ and we find integrating (A.4) over $\mathbf{R}^N\times\mathbf{R}^k$

$$\iint_{\mathbf{R}^N\times\mathbf{R}^k} f\log f\,dx\,dy \geq \iint_{\mathbf{R}^N\times\mathbf{R}^k} f(x,y)\,\{\log\rho(x)-\pi|y|^2\}\,dx\,dy\ .$$

Therefore,

$$\int_{\mathbf{R}^N}\rho(x)\log\rho(x)\,dx \leq \iint_{\mathbf{R}^N\times\mathbf{R}^k} f(\log f+\pi|y|^2)\,dx\,dy$$

and (A.3) is proven.

Appendix 2. An equivalent formulation of renormalized solutions

In this appendix, we show why the formulation of renormalized solutions of (VB) equations is in fact equivalent to the natural adaptation to the (VB) system of the formulation originally introduced in [25]. Indeed, we wish to show that if $f\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x,v}))$ $(f\geq 0)$ satisfies (A) and

$$\text{(A.5)}\qquad \frac{\partial\beta(f)}{\partial t}+\mathrm{div}_x\{v\beta(f)\}+\mathrm{div}_v\{F\beta(f)\}=\beta'(f)Q(f,f)\quad \text{in } \mathcal{D}'$$

for all $\beta\in C^1([0,\infty);\mathbf{R})$ such that $\beta'(t)(1+t)$ is bounded on $[0,\infty)$, then f is a renormalized solution of (VB) in the sense of section II. In fact, by a simple approximation argument (truncating β and g, smoothing g and β), we easily check it is enough to prove that (A.5) implies

$$\text{(A.6)}\qquad \begin{cases} \dfrac{\partial}{\partial t}\gamma(f-g)+\mathrm{div}_x\{v\gamma(f-g)\}+\mathrm{div}_v\{F\gamma(f-g)\} \\ =\gamma'(f-g)Q(f,f)-\gamma'(f-g)\left\{\dfrac{\partial g}{\partial t}+v\cdot\nabla_x g+F\cdot\nabla_x g\right\}\quad \text{in } \mathcal{D}' \end{cases}$$

for all $\gamma\in C_0^\infty([0,\infty);\mathbf{R})$, $g\in C_0^\infty(\mathbf{R}^{2N}_{x,v}\times[0,\infty))$. In order to do so, we first observe that $F\in L^\infty(0,T;W^{1,1}_{loc}(\mathbf{R}^N_x))$ $(\forall\, T\in(0,\infty))$ because of (A) and (42). Therefore, by the results of R.J. DiPerna and P.L. Lions [29], (A.5) implies

$$\text{(A.7)}\qquad \begin{cases} \dfrac{\partial}{\partial t}[\gamma(\beta_\varepsilon(f)-g)+\mathrm{div}_x\{v\gamma(\beta_\varepsilon(f)-g)\}+\mathrm{div}_v\{F\gamma(\beta_\varepsilon(f)-g)\}= \\ \gamma'(\beta_\varepsilon-g)\left\{\beta'_\varepsilon(f)\left\{Q(f,f)-\left[\dfrac{\partial g}{\partial t}+v\cdot\nabla_x g+F\cdot\nabla_v g\right]\right\}\right\}\quad \text{in } \mathcal{D}' \end{cases}$$

where $\beta_\varepsilon(t)=\dfrac{t}{1+\varepsilon t}$ on $[0,\infty)$ $(\varepsilon>0)$.

And (A.6) follows easily upon letting ε go to 0_+ provided we check that

$$\text{(A.8)}\qquad \gamma'(\beta_\varepsilon(f)-g)\beta'_\varepsilon(f)Q(f,f)\underset{\varepsilon}{\rightarrow}\gamma'(f-g)Q(f,f)\quad \text{in } L^1(\mathbf{R}^N_x\times K\times(0,T))$$

for all $T\in(0,\infty)$, compact set $K\subset\mathbf{R}^N_v$. Since the a.e. convergence is obvious, we only have to check that we have

(A.9) $$\left\{ 1_{(|\beta_\varepsilon(f)-g|\le C)} \frac{1}{(1+\varepsilon f)^2} Q^{\pm}(f,f) \text{ is relatively weakly compact in } L^1(\mathbf{R}^N_x \times K \times (0,T)) \right.$$

for all C, $T \in (0, \infty)$, compact set $K \subset \mathbf{R}^N_v$. And in view of (75), it is, as usual, enough to check (A.9) for Q^-. But, we have then a.e. on $\mathbf{R}^{2N}_{x,v} \times (0, \infty)$

$$1_{(|\beta_\varepsilon(f)-g|\le C)} \frac{1}{(1+\varepsilon f)^2} Q^-(f,f) \le 1_{(\beta_\varepsilon(f)\le C+g)} \frac{f}{1+\varepsilon f} L(f) \le CL(f)$$

since g is bounded. And we conclude.

CEREMADE
UNIVERSITÉ PARIS-DAUPHINE
PLACE DE LATTRE DE TASSIGNY
75775 PARIS CEDEX 16

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