THE RICCI CURVATURE EQUATION WITH ROTATIONAL SYMMETRY

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Introduction. In this paper, we study global solutions of the equation of prescribed Ricci curvature with rotational symmetry on balls and spheres. Despite many statements in this paper labelled “Proposition” and “Theorem”, the emphasis here is on phenomenology rather than theory—we are considering the simplest case of the nonlinear problem of finding a Riemannian metric with prescribed Ricci tensor, and immediately encounter an overwhelming variety of possible behavior. Even the two-dimensional case (for complete surfaces) is surprisingly subtle (see [CD]), and some of the insight we gained from studying surfaces will be useful in this paper.

We consider the standard action of the orthogonal group $SO(n)$ on the $n$-dimensional space $\mathbb{R}^n$ or the action of $SO(n)$ on the $n$-dimensional sphere $S^n$ which leaves the north (and south) poles of $S^n$ fixed. A symmetric covariant tensor of rank two on $\mathbb{R}^n$ or $S^n$ will be called rotationally symmetric if it remains invariant under the action of $SO(n)$. The standard Euclidean metric on $\mathbb{R}^n$ and the round metric on $S^n$ provide examples of such tensors. We denote the (doubly covariant) Ricci curvature tensor of any Riemannian metric $g$ by $\text{Ric}(g)$. We are interested in the following problem:

Problem. Suppose $T$ is a rotationally symmetric tensor on $\mathbb{R}^n$ or $S^n$. Is there a rotationally symmetric metric $g$ such that $\text{Ric}(g) = T$? If it exists, then to what extent is the solution of this problem unique?

Since the Ricci tensor is homothety-invariant, i.e., $\text{Ric}(cg) = \text{Ric}(g)$ for any positive constant $c$, we shall say that the solution of $\text{Ric}(g) = T$ is unique if any two solutions are homothetically related. A tensor is called nonsingular provided it induces at each point a linear isomorphism from the tangent to the cotangent space. As we shall see, for such tensors local existence and uniqueness results are reasonably easy to come by (Propositions 2.2 and 2.3).

There are two reasons that we only consider nonsingular 2-tensors $T$. First, uniqueness may fail to hold. In the nonrotationally-symmetric context, it is easy

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to concoct examples (think of Ricci-flat metrics) where the solution of $\text{Ric}(g) = T$ is not unique, even up to scaling (see [DK]). Second, even local existence may fail to hold. As shown in [D1], when the tensor $T$ is nonsingular, there always exists locally a metric $g$ with $\text{Ric}(g) = T$, but there are examples of singular $T$ for which there is no metric $g$ satisfying $\text{Ric}(g) = T$, even locally. Thus, to consider the global problem of prescribed Ricci curvature, it is reasonable to limit one’s attention to Ricci candidates $T$ that are nonsingular everywhere.

On a sphere or a ball, any nonsingular rotationally symmetric 2-tensor $T$ must be either positive definite or negative definite (Lemma 1.1). For such 2-tensors $T$, one might expect to have global existence of a rotationally symmetric metric $g$ with $\text{Ric}(g) = T$. However, to our surprise, despite the fact that local existence is guaranteed for such tensors, existence may fail to hold in the rotationally symmetric category on a domain (even a small ball) which is prescribed in advance, even without boundary conditions. We shall exhibit families of positive and negative definite rotationally symmetric 2-tensors $T$ none of which can be the Ricci tensor of any rotationally symmetric Riemannian metric on a fixed small ball (Examples 2.4 and 2.5).

In fact, the work of DeTurck and Koiso [DK] has already shown that for every positive definite tensor $T$ on a compact manifold $M$ without boundary, there is a constant $c_0(T)$ with the property that, if $c > c_0$, then $cT$ is not the Ricci tensor of any Riemannian metric on $M$ (a mirror image of this result in the rotationally symmetric category is given in Corollary 3.14 and Theorem 3.15). Our Example 2.4 indicates that, even for certain negative definite 2-tensors $T$, there might be an obstruction to the global existence of metric $g$ with $\text{Ric}(g) = T$. This problem appears complicated even for three-manifolds. On the other hand the problem of existence and uniqueness of metrics with prescribed Ricci tensors on compact two-dimensional manifolds has been settled in [D2].

At first sight, the problem of prescribing the Ricci tensor in the rotationally symmetric category appears to reduce to solving a system of two second-order ordinary differential equations in two unknowns. However, since rotationally symmetric metrics are globally conformally flat (Lemma 1.2), we can choose to normalize either the metric or the Ricci tensor in such a way that the normalized tensor contains only one function. It seems natural, then, to try to use a “clever” change of variables to reduce the size of the system we need to consider. To accomplish this, we introduce a geometrically natural scalar function $w_g$ which we call the Ricci potential. The equation satisfied by $w_g$ greatly simplifies the analysis of the equivariant Ricci equation $\text{Ric}(g) = T$ (see §1). It turns out that the single scalar equation for the Ricci potential is equivalent to the original Ricci system in the rotationally symmetric case when the Ricci candidate is nonsingular. Furthermore, this equation is first-order, and can be used to derive certain estimates for the unknown metric $g$. Using these estimates we can decide when the equivariant Ricci system $\text{Ric}(g) = T$ has a global rotationally symmetric solution. Moreover, the function $w_g$ contains useful information about the global
geometry of the solution manifold. We exploit this in the theorems and examples of §3, to determine whether the solution metric $g$ is complete or not, when it exists (Theorems 3.4, 3.7 and Proposition 3.13).

If $(\mathbb{R}^n, g)$ is a complete metric space and globally conformally diffeomorphic to the hyperbolic space, then $g$ is called conformally hyperbolic. If $(\mathbb{R}^n, g)$ is conformally diffeomorphic to Euclidean space, then $g$ is called parabolic. In some cases, we can decide in advance whether the solution $g$ of the Ricci system $\text{Ric}(g) = T$ will be parabolic or hyperbolic (Propositions 3.8 and 3.11).

A long-term goal of our work is to provide some useful tools to study the equivariant Ricci curvature system. The equivariant Ricci curvature system with other symmetry groups besides $SO(n)$ is still under investigation.

**Conventions.** When we are working on $\mathbb{R}^n$, we shall use the following conventions: We let $(t, \Theta)$ be polar coordinates on $\mathbb{R}^n$, so that $\Theta$ represents coordinates on the unit $(n-1)$-sphere $S^{n-1}$, and we let $d\Theta^2$ be the canonical metric on $S^{n-1}$. A rotationally symmetric tensor on $\mathbb{R}^n$ can then be expressed as $\alpha(t)dt^2 + \beta(t)d\Theta^2$, where $\alpha$ and $\beta$ are real-valued functions of $t$. We also define $B_3^2$ to be the set $\{(t, \Theta)|0 \leq t < \delta, \Theta \in S^{n-1}\}$, i.e., the ball of radius $\delta$ with respect to the usual Euclidean metric on $\mathbb{R}^n$. When we work on $S^n$, we treat it simply as $\mathbb{R}^n$, with appropriate boundary conditions at infinity, which are described in §3.

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1. A Potential Function for the Ricci Equation With Symmetry. In this section, we work only with rotationally symmetric metrics on $\mathbb{R}^n$. For such a metric, we shall introduce the scalar-valued Ricci potential function and determine the differential equation it satisfies.

First, however, we need some basic facts about rotationally symmetric tensors.

**Lemma 1.1.** Suppose $T = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2$ is a smooth, nondegenerate rotationally symmetric 2-tensor on $\mathbb{R}^n$. Then $T$ is either positive or negative definite everywhere. Furthermore, $\varphi(0) = \lim_{t \to 0} \psi(t)$.

**Proof.** Let $g_0$ be the standard Euclidean metric on $\mathbb{R}^n$, so that $g_0 = dt^2 + t^2d\Theta^2$, and let $V$ be any unit vector at the origin with respect to $g_0$. It is easy to see that $T(V, V) = \varphi(0)$, independent of the choice of the direction of $V$. This shows that $T$ is either positive or negative definite at the origin, depending on the sign of $\varphi(0)$. Since the eigenvalues of $T$ are all real, and are continuous functions on $\mathbb{R}^n$, their signs cannot change. Therefore, $T$ is either everywhere positive definite or everywhere negative definite, depending on $\varphi(0)$. To prove the limit equality, note that at any point $(t, \Theta)$ other than the origin, we can write $V = g_0(V, \frac{\partial}{\partial t})\frac{\partial}{\partial t} + g_0(V, t\frac{\partial}{\partial \Theta})t\frac{\partial}{\partial \Theta}$. If we choose $V$ to be a unit vector $g_0$-orthogonal to $\frac{\partial}{\partial t}$, then at such a point, $T(V, V) = \psi(t)$. If we parallel translate $V$ back to the origin along the radial geodesic, we obtain the limit equality.
LEMMA 1.2. Any smooth, rotationally symmetric metric \( g \) on \( \mathbb{R}^n \) can be expressed as

\[
g = e^{2f(t)}[(r'(t))^2 dt^2 + r^2(t) d\Theta^2],
\]

where \( r(t) \) satisfies \( r(0) = 0, r'(0) = 1 \) and \( r'(t) > 0 \) for all \( t \geq 0 \).

Proof. From Lemma 1.1, we know that we can write \( g = [h_1(t)]^2 dt^2 + t^2 [h_2(t)]^2 d\Theta^2 \), and that

\[
\lim_{t \to 0} \frac{1}{t} \left[ \frac{h_1(t)}{h_2(t)} - 1 \right] = 0.
\]

It therefore makes sense to set

\[
r(t) = t \exp \left( \int_0^t \frac{1}{\tau} \left[ \frac{h_1(\tau)}{h_2(\tau)} - 1 \right] d\tau \right),
\]

and then

\[
f(t) = \log [h_2(t)] - \log \left[ \frac{r(t)}{t} \right]
\]
to obtain (1).

From (1), we see that rotationally symmetric metrics on \( \mathbb{R}^n \) are conformally flat. We can use standard formulas for conformal changes of metric (e.g., [Be], p. 58) to calculate the Ricci tensor of a rotationally symmetric metric:

LEMMA 1.3. The Ricci tensor of the metric

\[
g = e^{2f(r)}[dr^2 + r^2 d\Theta^2]
\]
is

\[
\text{Ric} \,(g) = \alpha(r) dr^2 + r^2 \beta(r) d\Theta^2,
\]

where \( \alpha(r) = -(n - 1)[f_{rr} + f_r/r] \) and \( \beta(r) = -[f_{rr} + (2n - 3)f_r/r + (n - 2)(f_r)^2] \).

Of course, when we try to solve the Ricci system \( \text{Ric} \,(g) = T \) in the rotationally symmetric context, we are given \( T = \varphi(t) dt^2 + t^2 \psi(d\Theta^2) \), and we seek \( g \) in the form (1), where we do not know the function \( r(t) \) in advance. We conclude that the equation \( \text{Ric} \,(g) = T \) is equivalent in this case to

\[
- [f_{rr} + f_r]r'(t) = - \frac{\varphi(t)r(t)}{(n - 1)r'(t)}
\]

and

\[
- \left[ f_{rr} + \frac{2n - 3}{r} f_r + (n - 2)(f_r)^2 \right] r^2 = t^2 \psi.
\]
where \( f_r = f'(t)/r'(t) \) and \( f_{rr} = f''(t)/r'(t) \). Using (2), we can rearrange (3) as follows:

\[
\begin{align*}
t^2 \psi &= - \left[ f_{rr} + \frac{2n-3}{r} f_r + (n-2)f_r^2 \right] r^2 \\
&= -[rf_{rr} + f_r] r - (n-2)(2f_r r + (rf_r)^2) \\
&= - \left[ -\frac{\varphi}{n-1} \left( \frac{r}{r'} \right)^2 \right] - (n-2)[2f_r r + (rf_r)^2]
\end{align*}
\]

Thus, when \( \varphi \neq 0 \), we can conclude that (3) is equivalent to

\[
\frac{t^2 \varphi \psi}{n-1} = \left[ \frac{\varphi}{n-1} \left( \frac{r}{r'} \right)^2 \right] - (n-2)[2f_r r + (rf_r)^2].
\]

Now we are ready to consider the Ricci potential. If \( g \) is a rotationally symmetric metric on \( \mathbb{R}^n \), then every two-dimensional linear subspace \( W \subset \mathbb{R}^n \) which passes through the origin is a totally geodesic submanifold of \( (\mathbb{R}^n, g) \). Moreover, if \( W' \) is another such subspace, then \( W \) is isometric to \( W' \). Therefore, we consider a fixed two-dimensional totally geodesic subspace \( W \), and for \( x \in W \), we let \( K(x) \) be the Gauss curvature of \( W \) with respect to the metric induced from \( g \). We write \( dA \) for the induced area element, so that \( K dA \) is the curvature form of \( W \). With respect to a fixed polar coordinate system \( (t, \theta) \) on \( \mathbb{R}^n \) (where \( t \) need not be the distance from the origin with respect to \( g \)), we define the Ricci potential function \( \psi_g(t) \) by

\[
\psi_g(t) = \int_{D_t} K dA_g,
\]

where \( D_t = \{(s, \Theta) \in W|0 \leq s \leq t\} \) is a disk centered at the origin of \( W \subset \mathbb{R}^n \).

**Lemma 1.4.** If \( \psi_g \) is the Ricci potential of the metric \( g \) defined by (1), then \( \psi_g(t) = -t r(t) f_r(t) \) and

\[
\frac{d\psi_g}{dt} = \frac{\varphi r}{(n-1)r'}. \]

**Proof.** The restriction of the metric \( g \) to the plane \( W \) is of course given by \( g = e^{2f}[dr^2 + r^2 d\theta^2] \), where \( \theta \) is the restriction of \( \Theta \) to a great circle. A well-known formula for the Gauss curvature of a surface in polar coordinates yields

\[
K = -e^{-2f} \left[ f_{rr} + \frac{f_r}{r} \right].
\]

We can think of \( K \) as being a function of \( r \) or of \( t \), via (1). An easy computation
using (5) then shows that
\[
\frac{dw_g}{dt} = K(t)r(t)r'(t)e^{2f(t)}
\]
\[
= -[f_{rr}(t)r(t) + f_r(t)]r'(t)
\]
\[
= -\frac{d[f_r]}{dt}.
\]
Since \( w_g(0) = 0 = r(0) \), we get \( w_g(t) = -r(t)f_r(t) \) for all \( t \). This equation and (2) yields the second assertion of the Lemma.

Now we are ready to demonstrate that then the Ricci potential satisfies an uncoupled equation with coefficients involving only the components of the Ricci tensor.

**Proposition 1.5.** If \( g \) is a rotationally symmetric metric of class \( C^2 \), and if \( \text{Ric}(g) = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2 \) is everywhere nonsingular, then the Ricci potential \( w_g \) satisfies:

\[
\left( \frac{dw}{dt} \right)^2 = \frac{1}{n-1}[(n-2)\varphi(t)(w^2 - 2w) + t^2\varphi(t)\psi(t)]
\]

with initial conditions \( w(0) = w'(0) = 0 \). Moreover, \( w'(t)\varphi(t) > 0 \) for all \( t > 0 \).

**Proof.** This follows immediately from the definition (5) of \( w_g \), from Lemma 1.4, and from equation (4).

The converse of Proposition 1.5 will be our main tool for the study of existence of rotationally symmetric metrics with prescribed (nonsingular) Ricci tensors:

**Proposition 1.6.** If the rotationally symmetric 2-tensor \( T = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2 \) is nondegenerate everywhere, and if the ordinary differential equation (6) has a solution \( w \) which satisfies \( w(0) = w'(0) = 0 \) and \( w'(t)\varphi(t) > 0 \) for all \( t > 0 \), then the Ricci system \( \text{Ric}(g) = T \) is solvable. In fact, it has a solution \( g \) which is given explicitly by equation (1), where

\[
r(t) = t \exp \left( \int_0^t \left[ \frac{\varphi(\tau)}{(n-1)w'(\tau)} - \frac{1}{\tau} \right] d\tau \right),
\]

\[
f(t) = -\int_0^t w(\tau)\frac{r'(\tau)}{r(\tau)}d\tau + c
\]

and \( c \) is a constant.
Proof. This is an easy verification. If \( w \) satisfies (6) and \( r \) and \( f \) are given by (7) and (8), then one easily verifies that \( w(t) = -r(t)f_x(t) \), and that

\[
\frac{dw}{dt} = \frac{\varphi r}{(n-1)r'}.
\]

Since \( \varphi \neq 0 \), we can deduce from this that \( f \) and \( r \) satisfy (2) and (3). Therefore, \( g \) satisfies \( \text{Ric}(g) = T \).

2. Solving the Ricci System via the Potential Function. In this section, we use Propositions 1.5 and 1.6 and the ordinary differential equation (6) to study existence and uniqueness for the rotationally symmetric Ricci system. It is natural to begin with the question of local existence for nonsingular rotationally symmetric Ricci candidates \( T \). Since local existence of some Riemannian metric \( g \) satisfying \( \text{Ric}(g) = T \) was proved in \([D1]\), we point out that the issue here is the existence of a rotationally symmetric metric satisfying the equation.

**Lemma 2.1.** For any \( C^\infty \) nonvanishing functions \( \varphi \) and \( \psi \) satisfying \( \varphi(0) = \psi(0) \neq 0 \), there is a local (i.e., defined on some interval of the form \([0, \epsilon)\)) solution \( w(t) \) of the equation

\[
\left( \frac{dw}{dt} \right)^2 = \frac{1}{n-1}[(n-2)\varphi(w^2 - 2w) + t^2\varphi\psi]
\]

which satisfies the initial conditions \( w(0) = w'(0) = 0 \) and such that \( w'(t)\varphi(t) > 0 \) for all \( t > 0 \).

**Proof.** For \( n = 2 \), equation (6) reduces to \( w' = \pm t\sqrt{\varphi(t)\psi(t)} \), which clearly has (global) solutions, so we restrict our attention to the case \( n \geq 3 \). For simplicity, let

\[
G(t, w) = \frac{1}{n-1}((n-2)\varphi(t)[w^2 - 2w] + t^2\varphi\psi).
\]

We shall assume that \( \varphi(0) > 0 \) and consider the equation

\[
\frac{dw}{dt} = \sqrt{G(t, w)}
\]

with the initial conditions for \( w \) as specified in the statement of the Proposition. The situation here is complicated by the fact that \( \sqrt{G(t, w)} \) is not Lipschitz at \((t, w) = (0, 0)\), and moreover \( G(0, w) \) is \emph{negative} for small \( w > 0 \), so we are trying to solve equation (6) in a neighborhood of a singularity.

In order to circumvent these difficulties, we define the function

\[
H(t, w) = \max\{0, G(t, w)\}
\]
and consider the integral equation

\[ v(t) = \int_0^t \sqrt{H(s, \nu(s))} \, ds \]

Peano's theorem (see [H]) implies that equation (10) has a \( C^1 \) solution \( v \) which satisfies \( v(0) = v'(0) = 0 \) (the latter inequality because \( H(0,0) = 0 \)), and it is immediate that \( v'(t) \geq 0 \) so that \( v(t) \) is nondecreasing and nonnegative for \( t > 0 \). We claim further that \( v(t) > 0 \) for all \( t > 0 \); otherwise \( v \) would be identically zero on some interval \([0, t_0]\), which would further imply that

\[ 0 = H(t, 0) \geq \frac{1}{n-1} \varphi(t) \psi(t) \]

for all \( t \in [0, t_0] \), which would contradict the assumption that \( \varphi \psi > 0 \) everywhere.

To complete the proof, we will show that there exists a \( \delta > 0 \) such that \( G(t, v(t)) > 0 \) for all \( t \in (0, \delta) \). To prove this, note first that the connected component \( \Gamma \) of the zero set of \( G \) which contains the origin of the \((t, w)\)-plane is a curve which has equation \( w = u(t) \), where

\[ u(t) = 1 - \sqrt{\frac{(n-2) - t^2 \psi(t)}{n-2}}. \]

Our objective is therefore to show that \( v(t) < u(t) \) for \( t \in (0, \delta) \). We can choose \( \delta \) so that \( t^2 \psi(t) < n-2 \) and \( |t\psi'(t)| < |\psi(t)| \) for all \( t \in [0, \delta] \). This shows that \( u'(t) > 0 \) for \( 0 < t < \delta \), i.e., \( \Gamma \) is transverse to all horizontal lines. On the other hand, the solution \( v(t) \) of (10) is positive for \( t > 0 \), therefore there is a sequence \( \{t_i\} \) of positive values of \( t \), converging to zero such that \( v'(t_i) > 0 \) for all \( i \). Thus \( v(t_i) < u(t_i) \) for all \( i \). If there were a point \( \tau \in (0, \delta) \) such that \( v(\tau) \geq u(\tau) \), then \( v' \) would be zero in a neighborhood of \( \tau \), and therefore (since \( u \) is strictly increasing on \( (0, \tau) \), and so \( v(\tau) > u(t) \) for all \( t < \tau \)) \( v \) would be constant on \( (0, \tau) \). This contradiction shows that no such \( \tau \) can exist. We conclude that \( v \) is actually a solution of (10) on \([0, \delta]\), which finishes the proof of Proposition 2.1 in the case where \( \varphi > 0 \). One handles the other case by the same method.

The obvious corollary of Proposition 2.1 is the following.

**Proposition 2.2.** If \( T \) is a rotationally symmetric, nonsingular tensor, then the system \( \text{Ric}(g) = T \) has rotationally symmetric local solutions (in some neighborhood of the fixed point of the rotation group).

Our next concern is uniqueness:

**Proposition 2.3.** If the rotationally symmetric Ricci candidate \( T \) is nonsingular, then there is at most one rotationally symmetric metric (up to homothety) such that \( \text{Ric}(g) = T \).
Proof. From Propositions 1.5 and 1.6, we need only consider the question of uniqueness of solutions of equation (6). Again, we first consider the case \( \varphi > 0 \), and noting that \( w' \) has the same sign as \( \varphi \), namely positive, we restate equation (6) as

\[
\frac{dw}{dt} = \sqrt{G(t, w)}.
\]

Suppose there were two distinct solutions \( w_1 \) and \( w_2 \) of (11), each of which satisfied the initial conditions \( w_i(0) = w'_i(0) = 0 \) and the sign condition \( w_i(t) > 0 \) for \( t > 0 \) and \( i = 1, 2 \). It must therefore be true that \( G(t, w_i(t)) > 0 \) for all \( t > 0 \). Therefore, by the standard uniqueness theorem for ordinary differential equations [H], if \( w_1(t_0) = w_2(t_0) \) for some \( t_0 > 0 \) then \( w_1 \) and \( w_2 \) coincide for all \( t \). We can thus assume that

\[
w_1(t) > w_2(t)
\]

for all \( t > 0 \). But \( G \) is a decreasing function of \( w \) for small positive values of \( w \). This fact and (12) would imply

\[
w_1(\epsilon) = \int_0^\epsilon \sqrt{G(\tau, w_1(\tau)))}d\tau \leq \int_0^\epsilon \sqrt{G(\tau, w_2(\tau)))}d\tau = w_2(\epsilon)
\]

for small \( \epsilon > 0 \), which contradicts (12). Therefore we have uniqueness as stated. The proof for the case \( \varphi < 0 \) is essentially identical and is left to the reader.

We have reduced the question of finding rotationally symmetric solutions of \( \text{Ric}(g) = T \) to the question of existence of a solution \( w \) of the initial-value problem of Lemma 2.1. In the proof of Lemma 2.1, we saw that the question of the existence of \( w \) boils down to whether the graph of \( w \) avoids the region where the function \( G(t, w) \) of equation (9) is nonpositive. This region of the \( tw \)-plane is a band (sometimes of zero width), symmetric with respect to the line \( w = 1 \) (see Figure 1). The reason the local existence proof of Lemma 2.1 was somewhat delicate is that the initial conditions are just on the boundary of the band—but if \( \varphi(t)\psi(t) > 0 \) the solution curve manages to avoid the band for small \( t \).

We now turn to the question of the global existence of solutions of \( \text{Ric}(g) = T \). This can mean existence on all of \( \mathbb{R}^n \), or else on some specified ball \( B_e \subset \mathbb{R}^n \). Note that getting a solution on all of \( \mathbb{R}^n \) or on a specified ball is not as simple as a change of coordinates, because the given tensor \( T \) will “notice” the coordinate change. In a certain sense, we can think of \((B_e, \pm T)\) as a Riemannian manifold, and then the “size” of \( B_e \) takes on a coordinate-invariant meaning. As indicated above, to answer questions of global existence we will need to understand the interaction of the solution curve \( w = w(t) \) of equation (6) with the \( G \leq 0 \) band. We begin with some examples which show that the situation is indeed quite delicate.
Example 2.4. Let \( B_2 \) be the ball of (Euclidean) radius two in \( \mathbb{R}^n \) for \( n \geq 3 \), and let the negative tensor \( T \) be specified by

\[
T = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2,
\]

with

\[
\varphi(t) = -\frac{4(n-1)\pi^2}{n-2}
\]

for all \( t \), and choose \( \psi \) to be a smooth, negative function of \( t \) which satisfies

\[
\psi(t) = \begin{cases} 
-\frac{4(n-1)\pi^2}{n-2} & \text{if } 0 \leq t \leq \frac{1}{16} \\
-\frac{(n-2)}{t^2} & \text{if } \frac{1}{8} \leq t \leq 2
\end{cases}
\]

From Propositions 1.5 and 1.6, we know that finding a rotationally symmetric metric on \( B_2 \) with \( \text{Ric}(g) = T \) is equivalent to solving equation (6) on the interval \([0, 2]\), for a function \( w \) which satisfies \( w(0) = w'(0) = 0 \) and \( w \) is strictly decreasing for all \( t > 0 \). But for our choice of \( T \), on the interval \([\frac{1}{8}, 2]\), equation (6) becomes:

\[
w' = -2\pi \sqrt{2} - (1 - w)^2.
\]

All solution curves of this equation are unions of segments of the lines \( w = 1 + \sqrt{2} \), \( w = 1 - \sqrt{2} \), and a strictly decreasing part of the graph of

\[
w = 1 - \sqrt{2}\sin(2\pi t + c)
\]

(see Figure 2). Thus, if the solution curve \( w = w(t) \) has not encountered the \( G \leq 0 \) band for \( t < \frac{1}{8} \), then it is certain to do so for some \( t_0 < 2 \), and the solution metric will fail to exist for \( t > t_0 \) (since we must have \( w' < 0 \) for the metric, by Proposition 1.5). Thus there is no rotationally symmetric metric such that \( \text{Ric}(g) = T \) on \( B_2 \).

Figure 1.
It is interesting to ask why the solution fails to exist for $t > t_0$. Recall that we use $w$ and equations (7) and (8) to determine functions $r(t)$ and $f(t)$ so that the metric $g$ is of the form

$$g = e^{2f}(dr^2 + r^2d\Theta^2).$$

From the form of $w$ we see that as $t$ approaches $t_0$, $dw/dt$ approaches zero linearly, and so $r(t)$ approaches infinity by equation (7). Since $w$ is negative, $r$ is positive and $r'$ is positive, it follows from equation (8) that $f(t) \geq 0$ and $e^f \geq 1$ for all $t$. With respect to $g$, the distance $L$ from the origin to a point where $t = t_0$ is

$$L = \int_0^{t_0} e^{f(t)} \frac{dr}{dt} dt \geq \int_0^{t_0} \frac{dr}{dt} dt = \lim_{t \to t_0} r(t) = +\infty.$$

Thus, the Riemannian manifold $(B_{t_0}, g)$ (recall that $B_{t_0}$ is the ball of radius $t_0$ in $(t, \Theta)$ space) is complete. Even though the Ricci tensor extends smoothly beyond the “boundary” of the manifold, the metric cannot, since it is already complete. This situation is vaguely reminiscent of the Schwarzschild solution of Einstein’s equation in general relativity, where it appears that the solution has a singularity at some small radius, but the solution can in fact be continued inside this radius because the singularity is a result of the coordinates chosen, not the geometry. Here, however, it is the geometry and not the coordinates that goes wrong (or right, perhaps). Since the Riemannian manifold $(B_{t_0}, g)$ is (geodesically) complete, there can be no analog of the continuation of the Schwarzschild solution for our metric. We will have more to say about the completeness of our solutions in the next section.

**Example 2.5.** For an example with positive Ricci curvature, let $T = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2$ with

$$\varphi(t) = \frac{n - 1}{n - 2}.$$
and \(\psi\) a positive \(C^\infty\) function satisfying

\[
\begin{align*}
\psi(t) &= \frac{n-1}{n-2} \quad \text{if } 0 \leq t \leq \frac{1}{4} \\
\psi(t) &\leq \frac{3(n-2)}{4n^2} \quad \text{if } \frac{1}{4} \leq t \leq \frac{1}{4} \\
\psi(t) &= \frac{3(n-2)}{4n^2} \quad \text{if } \frac{1}{4} \leq t.
\end{align*}
\]

On the interval \(t \in [\frac{1}{4}, \infty)\), equation (6) becomes

\[
w' = \sqrt{(1 - w)^2 - \frac{1}{4}}.
\]

The \(G \leq 0\) region for \(t \geq \frac{1}{4}\) is the band between the lines \(w = \frac{1}{2}\) and \(w = \frac{3}{2}\), and all solution curves of this equation consist of portions of the lines \(w = 1/2\) and \(w = 3/2\) and of monotone increasing parts of the curves \(w = 1 + \frac{1}{2} \cosh(t + c)\) and \(w = 1 - \frac{1}{2} \cosh(t + c)\) (see Figure 3). Moreover, on any subinterval of \([0, 1/4)\) where \(w < 1/2\), we will have \(w' < 1\). Since \(w(0) = 0\), this guarantees that \(w(1/4) < 1/2\).

To ensure smoothness, the form of the solution for \(t > 1/4\) must therefore be \(w = 1 - \frac{1}{2} \cosh(t - t_0)\) for some \(t_0\) between 1/4 and 2. Therefore the solution curve will meet the \(G = 0\) curve, i.e., the line \(w = 1/2\) when \(t = t_0 \leq 2\). As before, the solution metric ceases to exist at that point. Using equations (7) and (8), we find that on the interval \((1/4, t_0)\) the metric takes the form \(g = e^{f(t)}(dr^2 + r^2 d\Omega^2)\) with

\[
\begin{align*}
f(t) &= -\frac{2}{n-2} \log (\csc(t-t_0) + \coth(t-t_0)) - \frac{1}{n-2} \log \sinh(t-t_0) \\
r(t) &= C(\csc(t-t_0) + \coth(t-t_0))^\frac{n-2}{n-2}
\end{align*}
\]

where \(C\) is a constant determined by the behavior of \(T\) on the interval \([0, 1/4)\).

For a radial geodesic, \(ds = e^f dr\), which behaves like \(1/(\sinh(t-t_0))^{\frac{n-2}{2}}\) as \(t\) approaches \(t_0\), thus the Riemannian manifold \((B_{t_0}, g)\) is complete, just as in Example 2.4.

By varying the parameters (i.e., the radii at which the function \(\psi\) changes, the precise values of \(\psi\), etc), we could construct a family of examples of this nature. We have thus produced the families of examples indicated in the introduction.

In a different direction, we can try to understand when the solution metric will exist on all of \(\mathbb{R}^n\), whether or not it is complete. The previous examples show that some condition must be imposed upon \(T\), and a closer examination of the proof of Lemma 2.1 yields the following sufficient condition:

**Theorem 2.6.** If the smooth, nonsingular, rotationally symmetric tensor \(T = \varphi(t) dt^2 + r^2 \psi(t) d\Omega^2\) satisfies

\[
\frac{d(t^2|\psi(t)|)}{dt} > 0
\]

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for all $t > 0$, then $\text{Ric}(g) = T$ has a rotationally symmetric solution defined on all of $\mathbb{R}^n$.

Proof. Recall that the solution of $\text{Ric}(g) = T$ depends upon our solving equation (6). As in the proof of Lemma 2.1, set $H(t, w) = \max\{0, G(t, w)\}$. Since $\sqrt{H}$ grows at most linearly in $w$, the initial-value problem

\begin{equation}
\frac{dw}{dt} = \sqrt{H(t, w)},
\end{equation}

$w(0) = 0$ will have a solution on all of $[0, \infty)$. We conclude that equation (6) will have a strictly monotonic solution on an interval as long as the solution curve $w = w(t)$ does not intersect the curve $G = 0$. The equation(s) of the $G = 0$ curve is

$$w = u(t) = 1 \pm \sqrt{1 - \frac{t^2 \psi}{n - 2}}.$$ 

Whether $T$ is positive or negative definite, we need only be concerned with the minus sign in this equation. In case $T$ is positive definite (so $\psi > 0$), we note that $u'(t) > 0$ on the curve where $G = 0$ provided $(t^2 \psi)' > 0$ (and the curve disappears when $t^2 \psi > n - 2$). But the mean-value theorem prohibits the solution curve of (6) from approaching the $G = 0$ curve from below but being horizontal at the point of contact. Thus, the solution curve does not meet the $G = 0$ curve, and so the solution of (13), which is defined on $[0, \infty)$ will also be a solution of (6). The case when $T$ is negative definite is proved in a similar way (see the proof of Theorem 3.7 in the next section).
Remark 2.7. In Examples 2.4 and 2.5, the Ricci candidates can be chosen to have the property that \( \frac{d(t^2 \psi(t))}{dt} \geq 0 \) for all \( t \), but of course \( \frac{d(t^2 \psi(t))}{dt} = 0 \) for all \( t \) beyond a certain value. Therefore, the assumption of Theorem 2.6 that this derivative is strictly positive is needed and in some sense sharp.

Remark 2.8. If \( w \) is a \( C^3 \) solution of equation (6), it is interesting to compute the second derivative of \( w \) at \( t = 0 \) (and compare it to the second derivative of \( 1 - \sqrt{1 - \frac{r^2 \psi}{n-2}} \), namely \( \frac{\psi(0)}{n-2} \)). Using equation (6), the initial conditions \( w(0) = w'(0) = 0 \), and the fact that \( \varphi(0) = \psi(0) \), we conclude that

\[
(w''(0))^2 = \lim_{t \to 0} \frac{w'(t)^2}{t^2} = \frac{\psi(0)}{n-1}(\psi(0) - (n-2)w''(0)).
\]

Since \( w \) and \( \psi \) must have the same sign, we must have \( w''(0) = \frac{\psi(0)}{n-1} \). This is another verification of the fact that, at least for small \( t > 0 \), the solution curve stays away from the \( G \leq 0 \) strip.

3. Completeness Issues. In this section, we study the question of whether or not there exists a complete rotationally symmetric metric which satisfies the equation \( \text{Ric}(g) = T \) on all of \( \mathbb{R}^n \). As indicated in the preceding section, there are two issues to deal with: First, the solution metric may not be defined on all of \( \mathbb{R}^n \), and second, even when the metric is so defined, it may or may not be complete.

Examples 2.4 and 2.5 showed that the relevant issue in proving completeness is to decide whether the distance from the origin to the "edge" of the solution manifold is infinite or not—and that the distance is given by \( \int e^f dr \), where \( r \) and \( f \) are defined in equations (7) and (8). The analysis of this integral in the two examples showed a distinction between the cases of positive and negative Ricci curvature—since \( f \) is positive when \( \text{Ric}(g) \) is negative and vice versa, the factor \( e^f \) in the distance integral is helping the metric to be complete in the negative case and hindering it in the former. Thus it should be easier to come up with necessary conditions for completeness in the positive Ricci case, and sufficient conditions in the negative case.

Metrics with Positive Ricci Curvature. Our initial observation is an immediate consequence of the reasoning in the previous paragraph.
PROPOSITION 3.1. (A necessary condition for completeness) Suppose $g = e^{2f(t)}(dr^2 + [r(t)]^2d\Theta^2)$ is a complete metric defined on a ball $B_{t_0} \subset \mathbb{R}^n$ with $\text{Ric}(g) \geq 0$, where $t_0 \leq \infty$. Then

$$\lim_{t \to t_0} r(t) = +\infty.$$

Proof. Let $L(t)$ be the distance from the origin to the boundary of the ball $B_t$ with respect to the metric $g$, i.e.,

$$L(t) = \int_0^t e^{\ell(s)}r'(s)ds.$$

Since $\text{Ric}(g) \geq 0$, the Ricci potential $w_g$ of the metric $g$ satisfies $w_g \geq 0$. Equation (8) shows that

$$f'(t) = -w(t) \frac{r'(t)}{r(t)} \leq 0$$

and so $L(t) \leq e^{f(0)}r(t)$. Therefore, if $g$ is a complete metric, we see that

$$\lim_{t \to t_0} r(t) \geq \lim_{t \to t_0} e^{-f(0)}L(t) = \infty.$$

This completes the proof of Lemma.

Remark 3.2. Lemma 3.1 is a very special case of the Ahlfors-Gromov Lemma (see [Gr], p. 198). Some assumption concerning the nonnegativity of the Ricci curvature is required, as evidenced by the example of the hyperbolic metric.

Together with Proposition 1.5, we can use Lemma 3.1 to give some conditions on $T$ which preclude the existence of a complete solution $g$ of $\text{Ric}(g) = T$ on all of $\mathbb{R}^n$.

Example 3.3. Suppose $\text{Ric}(g) = \varphi(t)(dt^2 + t^2d\Theta^2)$ with $\varphi'(t) \geq 0$ and $\varphi(t) \geq a > 0$, where $a$ is constant. If $w$ is the Ricci potential of $g$, then equation (6) implies

$$w'(t) \geq \sqrt{\frac{a}{n-1}}[at^2 + (n-2)((w-1)^2 - 1)].$$

Since $w(t)$ exists for all $t$ by Theorem 2.6, and since $w' \geq ct$ for $t$ sufficiently large, we see that $w \geq 1$ eventually. Thereafter, we can use (14) and Gronwall’s inequality to estimate

$$w(t) \geq [c_1e^{ct} + 1]$$
for some positive numbers $c_1$ and $c_2$. Using (14) again, we see that $w'$ also grows exponentially as $t \to \infty$. But then the integral for $r(t)$ in (7), namely

$$\int_0^t \left[ \frac{\varphi(s)}{(n-1)w'(s)} \right] ds$$

remains bounded for all $t > 0$. Therefore, $\lim_{t \to \infty} r(t)$ is finite and we conclude that the metric $g$ is not complete by Proposition 3.1.

The turning point of the proof of noncompleteness in Example 3.3 is the growth of $w$. Recall that $w$ is defined as

$$w(t) = \int_{D_t} K dA,$$

the integral of the Gauss curvature of a totally geodesic 2-disk passing through the center of symmetry. If $(\mathbb{R}^n, g)$ is a complete, rotationally symmetric manifold with positive Ricci curvature, it follows that the 2-planes through the origin of $\mathbb{R}^n$ are complete totally geodesic two-dimensional submanifolds of $(\mathbb{R}^n, g)$ with positive Gauss curvature. But Cohn-Vossen's inequality applied to any of these complete, positively curved surfaces then implies that we must have $w(t) \leq 1$ for all $t$ (and so we could have ended the proof of Example 3.3 four sentences sooner). This reasoning leads to another necessary condition for completeness:

**Theorem 3.4.** Suppose $\varphi > 0$ and $T = \varphi(t)(dt^2 + t^2d\Theta^2)$ satisfies

$$\int_0^\infty \max\{0, t^2\varphi^2(t) - (n - 2)\varphi(t)\} dt > \sqrt{n - 1},$$

then there is no rotationally symmetric, complete metric $g$ satisfying $\text{Ric}(g) = T$ on all of $\mathbb{R}^n$.

**Proof.** By the above reasoning, we need only show that $w(t) \leq 1$ for all $t$. But if this were so, we would have

$$1 \geq w(t) = \int_0^t w'(s)ds = \sqrt{\frac{1}{n-1}} \int_0^t \sqrt{s^2\varphi^2(s) + (n-2)\varphi(s)(w^2 - 2w)}ds$$

We let $t \to \infty$ and use the fact that $w^2 - 2w \geq -1$ to get

$$\int_0^\infty \max\{0, t^2\varphi^2(t) - (n - 2)\varphi(t)\} dt \leq \sqrt{n - 1},$$

which contradicts our assumption.
We can restate Theorem 3.4 in two-function form:

**Corollary 3.5.** If the positive-definite tensor \( T = \varphi(t)dt^2 + t^2d\Theta^2 \) satisfies
\[
\int_0^\infty \sqrt{\max \{0, t^2\varphi(t)\psi(t) - (n-2)\varphi(t)\}} dt > \sqrt{n-1},
\]
then there is no rotationally symmetric complete metric \( g \) which satisfies \( \text{Ric}(g) = T \) on all of \( \mathbb{R}^n \).

**Example 3.6.** Consider a paraboloid-like metric on \( \mathbb{R}^n \), for example,
\[
g = (4t^2 + 1)dt^2 + t^2d\Theta^2.
\]
A simple computation shows that
\[
\lim_{t \to \infty} w_g(t) = 1
\]
and \( \text{Ric}(g) > 0 \). When we express the Ricci tensor as \( \text{Ric}(g) = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2 \), we see that
\[
\int_0^\infty t\varphi(t)\psi(t)dt = 1.
\]
This shows that some assumption such as the one in Theorem 3.4 and Corollary 3.5 is needed, and in fact the estimate is sharp in the two-dimensional case.

**Metrics with Negative Ricci Curvature.** At the end of §2, we promised to study the completeness of the metrics on \( \mathbb{R}^n \) whose existence was proved in Theorem 2.6.

**Theorem 3.7.** If the negative-definite tensor \( T = \varphi(t)(dt^2 + t^2d\Theta^2) \) satisfies \( \frac{d(t^2\varphi(t))}{dt} < 0 \) for all \( t \), then there exists a complete metric \( g \), for which \( \text{Ric}(g) = T \) holds on all of \( \mathbb{R}^n \). Moreover, \( (\mathbb{R}^n, g) \) is conformally diffeomorphic to the standard Euclidean space \( E^n \), in other words \( (\mathbb{R}^n, g) \) is parabolic.

**Proof.** The existence of \( g \) is a consequence of Theorem 2.6. By the reasoning at the beginning of §3, it is sufficient to show that \( \lim_{t \to \infty} r(t) = \infty \) to prove both parabolicity and completeness. Recall that \( \varphi(t) \) and the Ricci potential \( w(t) \) of the solution metric are negative, and so by equation (6),
\[
w'(t) = -\frac{1}{n-1} [t^2(\varphi(t))^2 + (n-2)\varphi(t)(w^2 - 2w)] \geq \frac{t\varphi(t)}{\sqrt{n-1}}.
\]
Therefore, one has

\[ r(t) = r(t_0) \exp \left\{ \int_{t_0}^t \frac{\varphi(s)}{(n-1)w'(s)} \, ds \right\} \geq Ct \]

for some positive constant C. Therefore \( \lim_{t \to 0, 0} r(t) = \infty \) and we are done.

In fact, it was only to prove global existence that we used the hypothesis concerning \( \frac{d(t^2 \varphi(t))}{dt} \), so the proof of Theorem 3.7 actually yields the following.

**Proposition 3.8.** Let \( T = \varphi(t)(dt^2 + t^2 d\Theta^2) \) be negative definite, and suppose equation (6)

\[ w'(t) = \sqrt{\frac{1}{n-1}[(n-2)\varphi(t)(w^2 - 2w) + t^2 \varphi^2(t)]} \]

has a solution \( w \) with \( w(0) = w'(0) = 0 \) and \( w'(t)\varphi(t) > 0 \) for all \( t > 0 \). Then \( \text{Ric}(g) = T \) has a unique, complete parabolic, rotationally symmetric solution \( g \).

**Conformally Hyperbolic Metrics.** If the Riemannian manifold \((\mathbb{R}^n, g)\) is conformally diffeomorphic to standard \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \), then it is said to be **conformally hyperbolic**. When \( \text{Ric}(g) = T \) has a complete solution \( g \), it is natural to ask whether the metric \( g \) is conformally hyperbolic or parabolic. Furthermore, we would like to be able to tell the hyperbolicity of \( g \) in advance from the Ricci tensor \( T \).

In this section, we shall simplify certain computations by normalizing our nonsingular, rotationally symmetric Ricci candidates \( T \) in the same way as we did the metric in Lemma 1.2: we shall assume that \( T \), rather than being given in the form \( T = \varphi(t)dt^2 + t^2 \varphi(t)d\Theta^2 \), is given in the form

\[ T = \rho(s)(ds^2 + s^2 d\Theta^2) \quad (15) \]

where \( \rho(s) \) is defined on some interval \([0, s_0)\) with \( s_0 \leq \infty \).

If \( T \) had been given in the form \( T = \varphi(t)dt^2 + t^2 \psi(t)d\Theta^2 \) on the whole \( \mathbb{R}^n \) (i.e., if \( t \) ran from 0 to \( \infty \)), then recall from the proof of Lemma 1.2 that we can compute the upper limit \( s_0 \) for (15) via:

\[ s_0 = \exp \left\{ \int_{0^+}^\infty \frac{\varphi(t)}{t\psi(t)} \, dt \right\}. \]

**Example 3.9.** If the metric \( g \) is the standard hyperbolic metric defined on the ball \( B_1 = \{(s, \theta)|0 \leq s < 1\} \), i.e.,

\[ g = \frac{4}{(1 - s^2)^2}(ds^2 + s^2 d\theta^2) \]
then $\text{Ric}(g) = -(n-1)g$. Therefore, $s_0 = 1$ for this metric, and one has

$$\lim_{s \to 1} \rho(s) = -\infty.$$ 

This example leads us to make the following observation:

**Proposition 3.10.** If $T = \varphi(t)dt^2 + t^2\psi(t)d\Theta^2$, and if $\text{Ric}(g) = T$ has a complete, conformally hyperbolic solution $g$ defined on all of $\mathbb{R}^n$, then the following two conditions hold:

(i) For any $\epsilon > 0$ we have

$$\int_{\epsilon}^{\infty} \frac{\varphi(t)}{t\psi(t)} dt < \infty$$

(ii) $\lim_{t \to \infty} t\psi(t) = -\infty$.

In other words, for the normalized form $T = \rho(s)(ds^2 + s^2d\Theta^2)$, we have

(i) $s_0 < \infty$

(ii) $\lim_{s \to s_0} \rho(s) = -\infty$.

**Proof.** (i) Suppose $g$ is a complete solution of $\text{Ric}(g) = T$ on all of $\mathbb{R}^n$. If $s_0$ were infinite, then we would have a solution $w_g(s)$ of the equation

$$w'(s) = -\sqrt{\frac{1}{n-1}[(n-2)\rho(s)(w^2 - 2w) + s^2(\rho(s))^2]}$$

for all $s > 0$. But then Proposition 3.8 would imply that $g$ is parabolic, contrary to our assumption of hyperbolicity.

For (ii), we begin with our solution of equation (16) on the interval $0 < s < s_0 < \infty$. We will show that if $g$ is complete, and if $\lim \sup_{s \to s_0} \rho(s) = c > -\infty$, the $g$ must be parabolic. To begin, since $w$ satisfies equation (16) and is strictly decreasing, the graph of $w$ must lie strictly above the curve of zeroes of the right-hand side of the differential equation, i.e., above the curve

$$w = 1 - \sqrt{1 - \frac{s^2\rho(s)}{n-2}}.$$ 

Therefore

$$\lim_{s \to s_0} w(s) \geq \lim \sup_{s \to s_0} \left\{ 1 - \sqrt{1 - \frac{s^2\rho(s)}{n-2}} \right\} = 1 - \sqrt{1 - \frac{s_0^2c}{n-2}}.$$
which is finite. Now, as in the proof of Theorem 3.4, the restriction of a two-plane through the origin in $\mathbb{R}^n$ to the ball $B_{s_0}$ is a totally geodesic two-dimensional submanifold of $(B_{s_0}, g)$, and $\lim_{s \to s_0} w(s)$ is the total curvature of this surface. We have just shown that this total curvature is finite, and so, by a theorem of Milnor (see [M]), the manifold $(B_{s_0}, g)$ must be parabolic. This completes the proof of Proposition 3.10.

The same kind of reasoning leads to the following:

**Proposition 3.11.** If $T = \rho(s)(ds^2 + s^2 d\Theta^2)$ on $B_1$ with $0 > \rho(s) \geq -c \left(\frac{1}{1-s}\right)^{-\alpha}$ for some $\alpha > 0$ and $c > 0$, then the Ricci system $\text{Ric}(g) = T$ has no complete, conformally hyperbolic solution $g$ on $B_1$.

**Proof.** If there were such a complete metric $g$ defined on $B_1$, then its Ricci potential $w_g$ would have to satisfy

$$|w(s)| = \left| \int_0^s \sqrt{\frac{1}{n-1} \left[p(t)^2 + (n-2)p(t)(w^2 - 2w)\right]} dt\right|$$

$$= \int_0^s \sqrt{\frac{1}{n-1} \left[p(t)^2 - (n-2)|p(t)|(w^2 + 2|w|)\right]} dt$$

$$\leq \int_0^s \sqrt{\frac{1}{n-1} t^2(p(t))^2 dt}$$

$$\leq \int_0^s \frac{1}{\sqrt{n-1}} \frac{ct}{(1-t)^{1-\alpha}} dt < C,$$

where $C$ is a finite constant which is independent of $s$. Therefore, the total curvature of the two-dimensional totally geodesic submanifolds through the origin is finite, and so completeness would imply parabolicity by Milnor's theorem, as above.

**Metrics on the Sphere.** Some of the reasoning we have been using can be applied to prove nonexistence of global solutions of the Ricci system on the $n$-dimensional sphere. For simplicity, we restrict our attention to positive-definite Ricci candidates in the normalized form (15). This entails that the tensor $T$ in the form (15) be defined on all of $\mathbb{R}^n$, and the function $\rho(s)$ must decay like $1/s^4$ as $s \to \infty$.

**Example 3.12.** The round metrics on the sphere are given by

$$g = \frac{c}{(1+r^2)^2} (dr^2 + r^2 d\Theta^2).$$
All these metrics have the same Ricci tensor, namely
\[ \frac{4(n-1)}{(1+r^2)^2} (dr^2 + r^2 d\Theta^2). \]

Suppose we have been given a positive definite, rotationally symmetric tensor in the form (15) with the appropriate asymptotics, and suppose that a solution of equation (16) exists for all \( s \), which it turn gives rise to a metric \( g \) on \( \mathbb{R}^n \) via Proposition 1.6. The issue at hand is whether the metric has the right asymptotics to be defined “across infinity” on \( S^n \). To address this question, we consider two-dimensional totally geodesic submanifolds through the origin (which must close up to be 2-spheres). The Gauss-Bonnet theorem gives us that we must have \( \omega(s) \to 2 \) as \( s \to \infty \). But recall that the solution curve \( w = w(s) \) is trapped underneath the \( G(s, w) = 0 \) curve defined by equation (9), unless there are values of \( s \) over which there are no points of this curve, i.e., values of \( s \) for which \( s^2 \rho(s) > n - 2 \) (note that \( s = 1 \) is such a value for Example 3.12). If there are no such values of \( s \), then we must have \( w \leq 1 \) for all \( s \), and so the solution metric cannot come from a metric on \( S^n \). This yields

**Proposition 3.13.** Let \( T \) be a positive-definite, rotationally symmetric tensor defined on \( S^n \). In order for \( T \) to be the Ricci tensor of a metric on \( S^n \) it is necessary that \( \sup_{s \in [0, \infty)} s^2 \rho(s) > n - 2 \), where \( s \) and \( \rho \) are obtained by normalizing \( T \) as in equation (15).

This immediately implies a rotationally symmetric complement to the result of DeTurck and Koiso mentioned in the introduction:

**Corollary 3.14.** Let \( T \) be a positive-definite rotationally symmetric tensor on \( S^n \). There is a positive constant \( c_0 \) such that for \( c < c_0 \), there is no global, rotationally symmetric solution of \( \text{Ric}(g) = cT \) on \( S^n \).

**Example 3.12.** (continued). Let
\[ \frac{4/3(n-1)^2}{(1+s^2)^2} (ds + s^2 d\Theta^2), \]
so that \( T_1 \) is the Ricci tensor of the standard metric on \( S^n \). In [DK] it is shown that only the round metrics on \( S^n \) have \( T_1 \) for their Ricci tensor, and if \( \beta > 1 \), then there is no metric on \( S^n \) with Ricci tensor \( T_\beta \). From Proposition 3.13, we can easily compute that there is no rotationally symmetric metric on \( S^n \) with Ricci tensor \( T_\beta \) if \( \beta \leq (n-2)/(n-1) \).

We can fill in the gap between \( \frac{n-2}{n-1} \) and 1 if we work a little harder. The change of variables \( s \mapsto 1/s \) leaves the tensor \( T_\beta \) invariant, and leaves the “equator” \( s = 1 \) of the sphere invariant. Using this symmetry and the geometric definition of \( w \), we see that in order for a global solution of \( \text{Ric}(g) = T_\beta \) to exist
on $S^n$, it is necessary and sufficient that $w(1) = 1$ and $w'(1) > 0$. We can therefore prove that $\beta = 1$ is the only value of $\beta$ for which $\text{Ric}(g) = T_\beta$ is solvable by showing that $w_\beta(1)$ is a strictly increasing function of $\beta$ (where $w_\beta$ is the solution of equation (16) for $T_\beta$). Since the solution of the initial-value problem for (16) varies smoothly with the parameter $\beta$, the partial derivative of $w_\beta(s)$ with respect to $\beta$ satisfies the following differential equation:

$$
\frac{d}{ds} \left( \frac{\partial w_\beta}{\partial \beta} \right) = \frac{\partial}{\partial \beta} \sqrt{\frac{1}{n-1} \{ (n-2)\rho_\beta(s)(w_\beta^2 - 2w_\beta) + s^2 \rho^2_\beta(s) \}}
$$

$$
= \frac{n-2}{n-1} \frac{\rho_\beta(2w-2) \cdot \partial w_\beta}{\sqrt{G_\beta(s, w_\beta)}} \frac{\partial}{\partial \beta} + \frac{1}{2(n-1)} \frac{(n-2) \frac{\partial \rho_\beta}{\partial \beta} (w_\beta^2 - 2w_\beta) + 2s^2 \rho_\beta \frac{\partial \rho_\beta}{\partial \beta}}{\sqrt{G_\beta(s, w_\beta)}}
$$

with initial condition $\frac{\partial w_\beta}{\partial \beta}(0) = 0$. This is a linear equation, and it is easy to see that the sign of the solution is the same as the sign of the second term on the right-hand side of the equation. But this is the same as the sign of the numerator, which is

$$
(n-2) \frac{\partial \rho_\beta}{\partial \beta} (w_\beta^2 - 2w_\beta) + 2s^2 \rho_\beta \frac{\partial \rho_\beta}{\partial \beta} = \frac{n-1}{\beta} G_\beta(s, w_\beta) + \frac{1}{\beta} t^2 \rho^2_\beta > 0.
$$

Therefore $w_\beta(1)$ is strictly increasing with $\beta$, and so the equation $w_\beta(1) = 1$ can be true for only one value of $\beta$, namely $\beta = 1$.

We can do better still by using the recent result of Xu [X], who showed that the equation $\text{Ric}(g) = T$ can have at most one global solution (up to homothety) in each pointwise conformal class of metrics. Together with the above reasoning, this shows that for $\beta \neq 1$, the equation $\text{Ric}(g) = T_\beta$ has no solution of the form

$$
g = \rho(s, \Theta)(ds^2 + s^2 d\Theta^2)
$$

i.e., there is no solution of the problem even if we allow the function $\rho$ to depend on the angular coordinates.

The above reasoning can be easily applied to prove the following:

**Theorem 3.5.** Let $T$ be a positive definite rotationally symmetric tensor on $S^n$. There is at most one value of the constant $\beta$ for which the equation $\text{Ric}(g) = \beta T$ has a solution of the form (17).
REFERENCES


