Abelian and Tauberian Theorems: Philosophy

Bent E. Petersen

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Next quarter, in Mth 516, we will discuss the Tauberian theorem of Ikehara and Landau and its connection with the prime number theorem (among other things). The notion of a Tauberian theorem is not all that precise – it is described by a philosophy rather than a definition. Therefore in this notes I will present a discussion of the classification of some theorems as *Abelian* or *Tauberian* and present a few examples (without proofs) to give you a feeling for the concept.

The classification into Abelian (or direct) and Tauberian (or converse) theorems occurs in the following context: one has a map (usually linear with some continuity properties) between function spaces, $T : X \to Y$. An Abelian theorem is a theorem that deduces some property (usually asymptotic) of $T(f)$ from a property (usually asymptotic) of f.

A Tauberian theorem is a converse theorem: one deduces a property of f from a corresponding property of $T(f)$. One might object if T is injective there is no real difference between these classifications, but in practice the classification only arises in situations where the inverse of T lacks precisely the properties one might need for the desired conclusion. Thus extra hypotheses and delicate arguments are often needed for the Tauberian theorems.

Another consideration which tends to distinguish T from its inverse in this context is that T is frequently distinguished by tradition and popular usage. For example, we normally discuss the Laplace transform, and then its inverse, though there is no technical reason one could not reverse the order of presentation.

Finally let us note that Abelian theorems are hardly ever identified as such unless there is a corresponding Tauberian theorem, whereas Tauberian theorems are frequently identified without reference to any corresponding Abelian theorem.

Our story begins with Abel and the prototype of all Abelian theorems. Since divergent series and integrals often occur in practice, it is of interest to try to assign some meaning to some of them. There are many interesting ideas, for example principal values, finite parts, or summation techniques of Gauss, Weierstrass, Cesaro, Abel, Poisson, and so forth. One idea, due to Abel, is as follows: suppose (a_n) is a bounded sequence. Then $\sum a_n$ may diverge of course, but the power series $\sum a_n z^n$ has radius of convergence at least 1. Thus may we not assign some meaning to $\lim_{r\to 1}\sum_n a_n r^n$ provided this limit exists? And will not this limit have some relation to $\sum a_n$? When the limit exists it is called the Abel sum of the series $\sum a_n$.

As an example we see the Abel sum of the divergent series $\sum_n (-1)^n$ is $\frac{1}{2}$.

It is difficult to reconcile such ideas with Abel's assertion (1826),

> The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them one may draw any conclusion he pleases ...

except that later he notes of the conclusions drawn from arguments involving divergent series

> That most of these things are correct in spite of that is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question. (See [7].)

For the Abel summation to be a reasonable concept it ought to assign the ordinary sum to a convergent series. This leads us to Abel's theorem.

Theorem 1 (Abel, 1826). *Suppose $\sum_n a_n$ exists and is A. Then $\sum_n a_n z^n$ has radius of convergence ≥ 1 and*

$$\lim_{r\to 1}\sum_{n=1}^{\infty} a_n r^n = A.$$

Here our mapping T sends the sum $\sum_n a_n$ to the power series $\sum_n a_n z^n$. As an application of Abel's theorem we see that the sum of the alternating harmonic series $\sum_n (-1)^{n-1}/n$ is the natural logarithm of 2.

A proof of Abel's theorem may be found in Knopp's *little book on series* [8]. Note that r here approaches 1 along the real line. We can actually obtain the conclusion for nontangentail convergence in the disk – see, for example, Knopp's *big book on series* [9], item 233 in section 54. Abel's theorem in the form above is also proved in [9], item 100 in section 20, where in a footnote, Knopp states that Gauss stated and used Abel's theorem with nontangential convergence as early as 1812, but there was an error in Gauss' proof of it.

The prototype for all Tauberian theorems is Tauber's converse to Abel's theorem [12]:

Theorem 2 (Tauber, 1897). *If the power series $\sum_n a_n z^n$ has radius of convergence 1, if $\lim_{n\to\infty} n a_n = 0$, and if $\lim_{r\to 1}\sum a_n r^n = A$ then $\sum_{n=1}^{\infty} a_n = A$.*

In the presence of positivity we can weaken the hypotheses:

Theorem 3. *If the power series* $\sum_n a_n z^n$ *has radius of convergence* 1*, if* $a_n \geq 0$ *for each* n*, if* $\lim_{n\to\infty} a_n = 0$*, and if* $\lim_{r\to 1} \sum a_n r^n = A$ *then* $\sum_{n=1}^{\infty} a_n = A$*.*

A stronger version of Tauber's theorem is due to Littlewood:

Theorem 4 (Littlewood). *If the power series* $\sum_n a_n z^n$ *has radius of convergence* 1*, if there exists a constant* M *such that* $n\,|\,a_n\,| \leq M$ *for each* n*, and if* $\lim_{r\to 1} \sum a_n r^n = A$ *then* $\sum_{n=1}^{\infty} a_n = A$*.*

A proof of Littlewood's Tauberian theorem [11] based on Weiner's General Tauberian theorem [14] is given in [14], [2]. For Wiener's program to prove a great number of Tauberian theorems from the general one, see [13]. A nice presentation of Wiener's Tauberian theorem is given in Lars Gårding's beautiful book [3].

Wiener's General Tauberian theorem(s) follow from what Gårding [3] calls the Wiener *density theorem*:

Theorem 5 (Wiener 1932). *If* $K \in \mathcal{L}^1(\mathbb{R}^n)$ *and* M *is the linear span of all translates of* K*, then* M *is dense in* $\mathcal{L}^1(\mathbb{R}^n)$ *if and only if the Fourier transform* $\widehat{K}(\xi) \neq 0$ *for each* $\xi \in \mathbb{R}^n$*.*

It is pretty difficult to detect the Tauberian character of this theorem but the following consequence makes it clearer:

Theorem 6 (Tauberian Theorem, Wiener 1932). *Let* $K \in \mathcal{L}^1(\mathbb{R}^n)$ *satisfy* $\widehat{K}(\xi) \neq 0$ *for each* $\xi \in \mathbb{R}^n$*. Let* $f \in \mathcal{L}^\infty(\mathbb{R}^n)$ *and assume*

$$\lim_{|\,x\,|\to\infty} (K * f)(x) = A\widehat{K}(0)$$

exists (this equation then defines A*). Then for any* $G \in \mathcal{L}^1(\mathbb{R}^n)$ *we have*

$$\lim_{|\,x\,|\to\infty} (G * f)(x) = A\widehat{G}(0).$$

If we view the theorem as making an assertion about f the Tauberian character becomes clear.

Wiener's theorem may be used to prove the prime number theorem – the number of primes $\leq x$ is asymptotically equal to $x/\log x$, a result discovered by Gauss empirically in 1793 and proved in 1896 by Jacques Hadamard [4] and Charles-Jean del la Vallée Poussin [1]. A proof may also be based on the Ikehara Tauberian theorem [6] which is based on a theorem of Landau, [10]. See also Hardy–Littlewood [5], Wiener [14] and Donoghue [2].

Theorem 7 (Ikehara, 1931). *Let* μ *be a monotone nondecreasing function on* $(0,\infty)$ *and let*

$$F(s) = \int_1^\infty x^{-s}\, d\mu(x).$$

If the integral converges absolutely for $\Re s > 1$ *and there is a constant* A *such that*

$$F(s) - \frac{A}{s-1}$$

extends to a continuous function in $\Re s \geq 1$ *then*

$$\mu(x) \sim Ax.$$

The statement above is in terms of the Mellin transform – we may also state the theorem in terms of the Fourier–Laplace transform:

Theorem 8 (Ikehara, 1931). *Let* μ *be a monotone nondecreasing function on* $(0, \infty)$ *and let*

$$F(s) = \int_0^\infty e^{-sx} d\mu(x).$$

If the integral converges absolutely for $\Re s > 1$ *and there is a constant* A *such that*

$$F(s) - \frac{A}{s-1}$$

extends to a continuous function in $\Re s \geq 1$ *then*

$$\mu(x) \sim Ae^x$$

Here $\phi(x) \sim \psi(x)$ means asymptotic equality, that is, $\phi(x)/\psi(x)$ has limit 1 as $x \to \infty$.

I hope I have given you a bit of the flavor of what is a Tauberian theorem, but I do not want to leave you with the impression that all Tauberian theorems are deep and difficult. Here is a calculus Tauberian theorem that you may enjoy trying to prove as an exercise:

Exercise 1. Let f be a function on $[2, \infty)$ and suppose $xf(x)$ is monotone nondecreasing on $[2, \infty)$. Let m and n be real numbers, $n \neq -1$. If

$$\int_2^x f(t)dt \sim \frac{x^{n+1}}{(\log x)^m}, \quad x \to \infty,$$

then

$$f(x) \sim \frac{(n+1)x^n}{(\log x)^m}, \quad x \to \infty.$$

References

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Bent E. Petersen
Department of Mathematics
Oregon State University
Corvallis, OR 97331-4605

phone numbers
office (541) 737-5163
home (541) 753-1829
fax (541) 737-0517

petersen@math.orst.edu
http://ucs.orst.edu/~peterseb
http://www.peak.org/~petersen