Approximation Theory in Hilbert scales

Lecture Notes

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The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|...\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on K are:

i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$

ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,

iii) K is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} \quad .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} \quad .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$of eigenelements and eigenvalues

$K\varphi_i = \lambda_i \varphi_i$ with the properties

i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$

ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$

iii) the generalized Fourier sums $S_n := \sum_{i=1}^{n} (x, \varphi_i) \varphi_i \to x$ with $n \to \infty$ for all $x \in H$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|...\|$ and A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i \varphi_i \quad , \quad (\varphi_i, \varphi_k) = \delta_{i,k} \text{ and } \lim_{i \to \infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty} (x, \varphi_i) \varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2 .$$

Lemma: Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i) \varphi_i \quad , \quad \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2 \ ,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i) .$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{ \hat{x} | \hat{x} = (x_1, x_2, ...) \}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i) .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x},\hat{y})_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x},\hat{x})_\alpha$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite $\alpha-$ norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_α: they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x,y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x,x)_\alpha .$$

Because of the Parseval identity we have especially

$$(x,y)_0 = (x,y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax,Ax)_0 \quad , \quad H_2 = D(A) .$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $\lim \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, ...)$ with

$$\|\hat{x}\|_2^2 < \infty \quad , \quad \|\hat{x}\|_0^2 = \infty \quad .$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \quad \text{for} \quad x \in H_\gamma$$

with $\mu = \dfrac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \dfrac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \quad \text{and} \quad \|x\|_2 = \|Ax\|.$$

Extension and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x,y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x,\varphi_i)(y,\varphi_i)$$

$$\|x\|_{(t)}^2 = (x,x)_{(t)}^2 \quad .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i}t}$ instead of a polynomial decay in case of λ_i^{α}. Obviously we have

$$\|x\|_{(t)} \le c(\alpha,t)\|x\|_{\alpha} \text{ for } x \in H_{\alpha}$$

with $c(\alpha,t)$ depending only from α and $t > 0$. Thus the $(t)-norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_{\alpha}$ with $\alpha < 0$, is bounded by the $0-norm$ and the newly introduced $(t)-norm$. It holds:

Lemma: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0$, $\mu + \nu > 1$)

$$\|x\|_{\beta}^2 \le \nu\varepsilon\|x\|_{\gamma}^2 + \mu e^{-\nu/\mu}\|x\|_{\alpha}^2$$

applying Young's inequality to

$$\|x\|_{\beta}^2 \le (\|x\|_{\alpha}^2)^{\mu}(\|x\|_{\gamma}^2)^{\nu} \quad .$$

The counterpart of lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \le \|x\|$

ii) $\|y\|_1 \le \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \le e^{-t/\delta}\|x\|$.