

# A distributional way to prove the Goldbach conjectures leveraging the circle method

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## Abstract

Hardy-Littlewood [HaG] developed an analytical method, called *circle method*, which concerns with additive prime number problems. The circle method is about Fourier analysis over  $\mathbb{Z}$ , which acts on the circle  $\mathbb{R}/\mathbb{Z}$ . The analyzed functions are complex-valued power series

$$f(z) = \sum_0^{\infty} a_n z^n \quad , \quad |z| < 1.$$

The fundamental principle is ([Vil] chapter I, lemma 4, Notes)

$$r^n a_n = \int_0^1 f(re^{2\pi i t}) e^{-2\pi i n t} dt \quad , \quad 0 < r < 1.$$

The circle method is applied to additive prime number problems. Hardy-Littlewood [HaG] resp. Vinogradov [Vil] applied the Farey arcs resp. major and minor arcs ([HeH]) to derive estimates for corresponding Weyl sums ([WaA]) supporting attempts to prove the 2-primes resp. 3-primes Goldbach conjectures. All those attempts require estimates for purely trigonometric sums ([Vil]), as there is no information existing about the distribution of the primes, which jeopardizes all attempts to prove both conjectures.

This paper gives a conceptual design proposal to leverage the circle method to prove both Goldbach conjectures: it is proposed to replace the discrete Fourier transformation applied for power functions  $f(x)$  by continuous Hilbert- ( $H$ ), Riesz- ( $A$ ) resp. Calderon-Zygmund-transformations ( $S$ ) (which are Pseudo Differential Operators of order  $0$ ,  $-1$  and  $1$ ) with distributional, periodical Hilbert space domains  $H_{\alpha}^{\#}(0,1)$ . The analogue fundamental principle is

$$-n f_n(x) = \frac{1}{2} \int_0^1 \frac{f_n(y)}{\sin^2(\pi(x-y))} dy =: [Sf_n](x) = [A^{-1} f_n](x)$$

for

$$f_n(y) := a_n \cos 2\pi n y + b_n \sin 2\pi n y \quad .$$

This enables a discrete wavelet analysis (e.g. [GoJ], [LoA]) of a distributional prime number density function on  $\Gamma := S^1(\mathbb{R}^2)$ , resp. of the distributional von Mangoldt function ([ViJ])

$$\psi'(x) = \sum_{n < x} \Lambda(n) \delta(x-n) \in H_{-1/2-\varepsilon}(\mathbb{R}) \quad .$$

The distributional fractional part and  $\cot(\pi x)$ -functions ensure the link to the Zeta function in the Hilbert space framework  $H_{\alpha}^{\#}(0,1)$ . As mother function we propose the Kummer function (resp. its derivative)

$$b(x) := {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, 2\pi i x\right)$$

with its relationship to the  $erf(x)$  function and its zeros  $x_n \in (n-1/2, n)$ . The translation and scaling of the daughter wavelets are to be defined properly by the parameters  $n$  and  $p_n$ , which is not in scope of the paper.

## Baseline and Solution Approach

The content of this section is an extract from [BrK1] [Vi11], [WoD].

The Hardy-Littlewood circle method ([HaG]) has been applied in additive number theory e.g. for partition problems, i.e. an analysis about the number of additive partitions of  $N$  in a given number  $\nu$  of integer summands.

The (3-primes, 2-primes) Goldbach conjectures are concerned with  $\nu = 2, 3$ .

The method is based on the generating power series for the prime number sequence series  $P$

$$F(z) = \sum_{p \in P} z^p, \quad |z| < 1, \quad z \in \mathbb{C}.$$

Depending from the number  $\nu = 2, 3$  of summands for  $0 < r < 1$  it's about an analysis of

$$R_\nu(N) = \frac{1}{2\pi i} \int_{|z|=r} F^\nu(z) z^{-N} \frac{dz}{z} = r^{-N} \int_0^1 \left( \sum_{p \in P} r e^{2\pi i p \theta} \right)^\nu e^{-2\pi i N \theta} d\theta$$

fulfilling according to the Cauchy integral theorem

$$F^\nu(z) = \sum_{N \in \mathbb{N}} R_\nu(N) z^N.$$

The key principle of the circle method is the fact, that for  $N$  being an integer it holds

$$\int_0^1 e^{2\pi i N \alpha} d\alpha = \begin{cases} 1 & \text{if } N = 0 \\ 0 & \text{otherwise} \end{cases}$$

which can be reformulated in the form

**Lemma:** Let  $f(x) = \sum_0^\infty a_n x^n$  with  $|x| < 1$ , then for  $0 < r < 1$  it holds

$$r^n a_n = \int_0^1 f(re^{2\pi i t}) e^{-2\pi i n t} dt.$$

The corresponding properties for our approach ([BrK1]) to the above lemma are captured in

**Lemma:** Let  $f_n(y) := a_n \cos 2\pi n y + b_n \sin 2\pi n y$ , then it holds for  $x \in (0, 1)$

$$-n f_n(x) = \frac{1}{2} \int_0^1 \frac{f_n(y)}{\sin^2(\pi(x-y))} dy =: [Sf_n](x) = [A^{-1} f_n](x).$$

We note that

$$|1 - e^{2\pi i \alpha}|^2 = 4 \sin^2(\pi \alpha).$$

**Lemma:** Let

$$b(x) := {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; 2\pi i x\right)$$

denote the Kummer function with its zeros  $x_n \in (n-1/2, n)$  and  $\Gamma := S^1(R^2)$ . Then the transformation into an appropriate distribution Hilbert space framework is enabled by the following three isometric mappings:

$$\begin{aligned} e: R &\rightarrow \Gamma, & e: x &\rightarrow e^{2\pi i x} \\ j: \Gamma &\rightarrow \Gamma, & j: e(x_n) &\rightarrow e(p_n) \\ k: N &\rightarrow R, & k: n &\rightarrow x_n \end{aligned}$$

The circle method is about Fourier analysis over  $Z$ , which acts on the circle  $R/Z$ . Fourier analysis builds on periodical functions. It provides frequency-domains for corresponding analysis of the frequency of the original function. Fourier inverse then allows conclusions back to the origin function. The counterpart of FT to analyze non-periodical function is wavelet transforms, which is proposed to replace the arcs concept. The proposed corresponding Hilbert space framework is given by  $H_\alpha^\#(0,1)$  ( $\alpha \leq 0$ ). The relationship to prime number theorem is given by ([ViJ])

$$\psi(x) := \sum_{p^* < x} \log p = \sum_{n < x} \Lambda(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left[ -\frac{\zeta'(s)}{\zeta(s)} \right] x^s \frac{ds}{s}, \quad J(x) = \int_0^\infty \frac{d\psi(x)}{\log x}$$

with

$$\psi'(x) = \sum_{n < x} \Lambda(n) \delta(x-n) \in H_{-1/2-\epsilon}(R)$$

and

$$H\psi'(x) = \sum_{n < x} \Lambda(n) H\delta(x-n) = \sum_{n < x} \frac{\Lambda(n)}{\pi(x-n)} \in H_{-1/2-\epsilon}(R).$$

The relationship to the prime number distribution is given by the fractional function ([TiE]) resp. its Hilbert transform

$$\begin{aligned} \varphi(x) &:= 2\pi(x - [x]) = \pi - \sum_1^\infty \frac{2}{n} \sin 2\pi n x \\ \varphi_H(x) &:= -2 \log 2 \sin(\pi x) = \sum_1^\infty \frac{2}{n} \cos 2\pi n x \end{aligned}$$

fulfilling the following properties

**Lemma:** In a distributional sense with appropriate domains  $D(A) \subset H_\alpha^\#(0,1)$ ,  $D(S) \subset H_{\alpha+1}^\#(0,1)$  it holds

$$\varphi_H(x) + i\varphi(x) = \sum_1^n \frac{2}{n} [\cos 2\pi n x + i \sin 2\pi n x] = A[\sigma_H + i\sigma](x) \in L_2^\#(0,1)$$

$$\sigma_H + i\sigma = \sum_1^n \frac{n}{2} [\cos 2\pi n x + i \sin 2\pi n x] = S[\varphi_H + i\varphi](x) \in H_{-1/2}^\#(0,1)$$

i.e.

$$(\varphi_H + i\varphi, \chi) = (A[\sigma_H + i\sigma], \chi) \quad , \quad \forall \chi \in L_2^\#(0,1)$$

$$(\varphi_H + i\varphi, \chi)_{-1/2} = (S[\sigma_H + i\sigma], \chi)_{-1/2} \quad , \quad \forall \chi \in H_{-1/2}^\#(0,1)$$

**Lemma:** For  $0 < \text{Re}(s) < 1$  corresponding Mellin transforms are given by

$$\zeta(s) = -(-s) \int_0^\infty x^{-s} ([x] - x) \frac{dx}{x} = -M \left[ x \frac{d}{dx} ([x] - x) \right] (-s)$$

$$\zeta(s) \Gamma(s) \cos\left(\frac{\pi}{2} s\right) = M[\varphi_H](s) = \zeta(s) M[\cos](s) \quad \text{in a weak } L_2^\#(0,1) \text{ - sense}$$

$$M[\sin](s) = \Gamma(s) \sin\left(\frac{\pi}{2} s\right) \quad , \quad M[\cos](s) = \Gamma(s) \cos\left(\frac{\pi}{2} s\right) \quad .$$

From [LaE] we recall:

Let  $G_n$  denote the number of all partitions of the even integer  $n$  in a sum of two primes  $p$  and  $q$  (whereby  $p+q$  and  $q+p$  are counted as two different partitions) and  $H(x)$  the number of prime pairs, for which  $p+q \leq x$  , then it holds

$$H(x) = \sum_{p \leq x} \pi(x-p) \approx \int_2^{x-2} \pi(x-u) \frac{du}{\log u} \approx \int_2^{x-2} \frac{x-u}{\log(x-u)} \frac{du}{\log u} \quad .$$

Every  $n$  can be represented as sum  $n_1 + n_2$  in  $n-1$  different way. According to the prime number theorem the probability that a selected integer is prime is about  $\approx 1/\log n$ . Therefore an even  $n$  seems to have  $\approx n/\log n$  sums of two primes. In view of estimates of

$$S(x) = \sum_0^{m-1} e^{2\pi i \alpha x / m}$$

we recall from [Vil] chapter 1,

**Lemma:** Let  $F(x) = P(x) + iQ(x)$  be a periodic function of  $x$  with period 1, and suppose that the interval  $0 < x \leq 1$  can be split up into finite number of intervals, such that the real functions  $P(x)$  and  $Q(x)$  are continuous and monotonic in the interior of each. Suppose further that

$$F(x) = \frac{1}{2} [F(x+0) + F(x-0)]$$

at each point of discontinuity of the function. Then

$$F(x) = \frac{1}{2} a_0 + \sum_1^\infty a_n \cos 2\pi n x + b_n \sin 2\pi n x$$

with

$$a_n = 2 \int_0^1 F(\xi) \cos 2\pi n \xi d\xi \quad \text{and} \quad b_n = 2 \int_0^1 F(\xi) \sin 2\pi n \xi d\xi \quad .$$

**Remark:** A sufficient condition that the Fourier series converge is the Dirichlet condition:

The interval  $(0,1)$  is the union of finite intervals, where the function  $F(x)$  is continuous  
 For all points  $x_s$  where  $F(x)$  is non continuous,  $F(x_s + 0)$  and  $F(x_s - 0)$  exist, then it holds

$$\frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos 2\pi nx + b_n \sin 2\pi nx = \begin{cases} F(x) & \text{if } F(x) \text{ continuous} \\ \frac{1}{2}[F(x+0) - F(x-0)] & \text{else} \end{cases}$$

**Remark:** The analysis of the fractional part function and its finite partial sums leads to the Gibbs phenomenon ([GrT]):

the finite part sums have maxima and minima in the neighborhood of  $x_s$ , but the Fourier series above is divergent at  $x_s$ . Those sums are examples of continuous functions, where its corresponding Fourier series is divergent at certain points. The situation is different in case of the lower regularity assumption  $F \in H_{-1/2}^{\#}(0,1)$ .

From [EdH] 1.11, 3.7, we recall Riemann's prime number distribution function  $J(x)$  definition

$$J(x) := \frac{1}{2} \left[ \sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right]$$

and a special representation ( $r = 1$ ) of the von Mangoldt function with the zeros of the Zeta function in the form

$$x \int_0^x \frac{d\psi(t)}{t} = \sum_{\rho} \frac{x^{\rho}}{1-\rho} + \sum_1^{\infty} \frac{x^{-2n}}{2n+1} \cdot$$

It holds

$$J(x) = \int_0^x \frac{d\psi(t)}{\log t} = \sum_{n < x} \frac{\Lambda(n)}{\log} = \int_0^{\infty} \sum_{n < x} \frac{\Lambda(n)}{n^r} dr \cdot$$

In a distribution framework the analogue of e.g.  $J'(x)$  is  $S[\tilde{J}](x) \in H_{\alpha}$ , which allows reduced regularity assumptions to the domain, than the Dirichlet conditions.

**Remark:** With respect to additive number theory problems and an alternative density definition we also refer to Schnirelmann density ([PeB]).

With respect to the Mellin and the Hilbert transforms of the Gaussian and Kummer function

$$G(x) := e^{-\pi x^2} \quad , \quad K(x) := {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -\pi x^2\right)$$

we recall from [BrK1]:

**Lemma:** It holds

- i.)  $G_H(x) = 2xG(x)K(ix)$
- ii.)  $K(x) = \frac{1}{x} \int_0^x G(t)dt \quad \text{resp.} \quad dK = \frac{G(x) - K(x)}{x} dx$
- iii.)  $\int_0^\infty x^{s/2} dK = \frac{\Gamma(1+s/2)}{s-1}$  in the critical stripe
- iv.)  $M[G_H](s) = \frac{1}{2} \pi^{-s/2} \frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}$ .

**Lemma:** For the Hilbert transform  $G_H$  of the Gaussian function  $G$  it holds

- i.)  $G_H(x) = 4\pi x G(x) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \pi x^2\right) = 2x \sum_{n=0}^\infty (-1)^n \frac{(2\pi)^{n+1}}{1 \cdot 3 \cdot \dots \cdot (2n+1)} x^{2n}$
- ii.) The functions  $G_H$  and  $G$  are identical in a weak  $L_2(-\infty, \infty)$ .

**Remark:** The theorem of Erdős-Kac ([ErP]) concerning the Gaussian law of errors in the theory of additive number theoretic functions gave a first linkage from probability theory and additive number theory.

**Remark:** From ([BeB] 8, Entry 17 (iv), Entry (v)) we quote

*“Ramanujan informs us to note that*

$$\sigma(x) := \cot(\pi x) = 2 \sum_{v=1}^\infty \sin(2\pi v x) \quad ,$$

*which also is devoid of meaning, .... may be formally established by differentiating the equality*

$$2 \sum_{v=1}^\infty \frac{\cos 2\pi v x}{v} = -2 \log 2 \sin(\pi x)$$

*and that for*

$$\varphi(x) := \sum_{k=1}^x \frac{\log k}{k}$$

*it holds (“for the same limits”)*

$$\varphi(x-1) - \varphi(-x) = \pi(\gamma + \log(2\pi) \cot(\pi x) + 2\pi \sum_{k=1}^\infty \sin(2\pi k x) \log k$$

In the  $H_{\alpha}^{\#}(0,1)$  – framework it holds (see also [ZyA] XIII, (11-3))

$$(A\sigma)(x) := \sum_{-\infty}^{\infty} \frac{1}{2|n|} \sigma_n(x) = \sum_1^{\infty} \frac{\sin(2\pi nx)}{n} \in H_0^{\#}$$

i.e.  $\sigma \in H_{-1/2}^{\#}(0,1)$  in a weak sense. The corresponding Hilbert and Calderon Zygmund Pseudo Differential operators (of order 0 resp. 1) on the unit circle are given by

$$Au(\varphi) := \frac{1}{2\pi} \oint \ln \frac{1}{2 \sin \frac{\varphi - \vartheta}{2}} u(\vartheta) d\vartheta$$

$$Hu(\varphi) := \frac{1}{\pi} \oint \frac{1}{2} \cot \frac{\varphi - \vartheta}{2} u(\vartheta) d\vartheta$$

$$Su(\varphi) := \frac{1}{\pi} \oint \frac{u(\vartheta)}{4 \sin^2 \frac{\varphi - \vartheta}{2}} d\vartheta \quad .$$

The Hilbert space is linked to the Dirchlet series by the following ([LaE1] §227, Satz 40):

**Theorem:** The Dirichlet series

$$f(s) := \sum_1^{\infty} a_n e^{-s \log n} \quad g(s) := \sum_1^{\infty} b_n e^{-s \log n}$$

are convergent for  $s > -\varepsilon$  ( $\varepsilon > 0$ ). Then on the critical line it holds

$$((f, g))_{-1/2} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(1/2 + it) g(1/2 - it) dt = \sum_1^{\infty} \frac{1}{n} a_n b_n \quad .$$

The theorem is applied to derive Dirichlet series for  $\frac{1}{\zeta(s)}$  and  $\frac{\zeta(2s)}{\zeta(s)}$ .

With respect related functions (e.g. von Mangoldt function) we use the following notations

$$J(x) \rightarrow J_*(x), \quad \psi(x) \rightarrow \psi_*(x), \quad Li(x) \rightarrow Li_*(x) \quad .$$

As in the proposed framework the derivative operator is replaced by the Calderon-Zygmund operator with reduced domain regularity) this leads to the alternative density functions

$$J'(x) \rightarrow S[J_*](x), \quad \psi'(x) \rightarrow S[\psi_*](x) \quad .$$

Minor arcs are defined as the complementary set of the union of major arcs with respect to  $[0,1]$ . In [HaG] only major (Farey) arcs are applied with center  $a/q$  and  $q \leq N^{1/2}$ . We note that

$$\prod_{p < \sqrt{n}} \left(1 - \frac{1}{p}\right) \approx 2 \frac{e^{-\gamma}}{\log n} \quad .$$

The Fourier transformation provides (“just and only) frequency-domains in case of to be analyzed periodical function. To analyze non-periodical functions an appropriate transformation needs to have two properties: scaling and translation ([LoA]). This is given by the wavelet transform. Scaling a mother wavelet is the process of compressing or expanding the wave. Translation is moving the

wavelet along the inputted signal. The amount of translation that needs to be done depends on the scale of the wavelet. The smaller the scale the more translation that needs to occur.

We note that e.g. the analogue Mexican hat mother wavelet has the form ([LoA] (1.1.3))

$$\chi(x) := \frac{1}{2\pi} \frac{d^2}{dx^2} \left[ x \Phi' \left( \frac{x}{2\pi} \right) \right].$$

We further note that

the Hilbert transform preserves orthogonality of translates and scaling relations ([WeJ]). The Hilbert scale factor  $\alpha = -1/2$  of  $H_{-1/2}^\#(0,1)$  is related to the wavelet theory ([GoJ], [LoA]) on the unit disk (e.g. Daubechies and Möbius wavelets)

the reproducing property (Calderón's reproducing formula for the continuous wavelet transform) of Möbius wavelets is valid in a weak  $H_{-1/2}$ -sense.

The working assumption is that the arc analysis of the Weyl sums can be replaced by discrete wavelet analysis

$$\theta_*^{m,k}(x) := 2^{-m/2} \theta_*(2^{-m}x - k)$$

in a  $H_\alpha^\#(0,1)$ -framework with appropriately defined scaling and translation factors related to  $n, p_n$ .

The Kummer function ([AbM] 7.1.6)

$$K(x) = \frac{1}{x} \int_0^x G(t) dt = e^{-\pi x^2} \sum_0^\infty \frac{(2\pi)^n x^{2n}}{1 \cdot 3 \cdot \dots \cdot (2n+1)} =: \frac{1}{x} \hat{g}(x)$$

with

$$K'(0) = 0 \text{ and } G(x) = ixg(x)$$

is proposed to define an appropriate wavelet function with respect to  $H_\alpha(-\infty, \infty)$ -framework.

We note the series representation of the Kummer functions

$${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, x\right) = \sum_0^\infty \frac{1}{2n+1} \frac{x^n}{n!} \quad \text{and} \quad K_2(x) = x^{-1/2} {}_1F_1\left(0; \frac{1}{2}, x\right) = \sum_0^\infty \frac{x^{n-1/2}}{2n+3} \left[ \frac{1}{2} \cdot \frac{3}{2} \cdot \dots \cdot \left(n - \frac{1}{2}\right) \right]^{-1}$$

which are the solutions of the Kummer differential equation

$$L[u](x) := xu'' + \left(\frac{3}{2} - x\right)u' - \frac{1}{2}u = 0 \quad .$$

With the notation from [NaS] it holds  $xK'(x) \in l_2^0$  and therefore  $K'(x) \in l_2^{-1/2}$  and  $K(x) \in l_2^{1/2}$ .

For the expansion of the Kummer function in terms of Laguerre polynomials and Fourier transforms we refer to [PiA].

For an evaluation of the integral



$$2 \int_0^{\infty} G(t) \int_t^{\infty} \frac{\cos(\xi x)}{x} dx dt$$

we refer to [Gr] 4.229, 8.232 with the formulas

$$\begin{aligned} ci(x) &:= -\int_t^{\infty} \frac{\cos(x)}{x} dx = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t} dt = \gamma + \log x + \sum_1^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!} \\ &= \int_0^1 \log(\log \frac{1}{t}) dt + \log x + \sum_1^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!} . \end{aligned}$$

We further note the identities

$$ci(x) \pm si(x) = Ei(\pm ix)$$

and

$$C(z) + iS(z) = {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; i\frac{\pi}{2}z^2\right)$$

for the Fresnel integrals ([AbM] 7.3)

$$\begin{aligned} C(z) &:= \int_0^z \cos\left(\frac{\pi}{2}t^2\right) dt = \sum_0^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!(4n+1)} z^{4n+1} = -C(-z) \xrightarrow{z \rightarrow \infty} \frac{1}{2} \\ S(z) &:= \int_0^z \sin\left(\frac{\pi}{2}t^2\right) dt = \sum_0^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!(4n+3)} z^{4n+3} = -S(-z) \xrightarrow{z \rightarrow \infty} \frac{1}{2} . \end{aligned}$$

The conceptual idea is to transfer the Hardy-Littlewood method and its underlying framework into a distributional Hilbert space framework to enable approximation theory in the framework of functional analysis.

**Definition:** The function  $e(x)$  defines an isomorphism between  $R$  and the boundary of the unit circle  $\Gamma := S^1(R^2)$ . We denote with  $\tilde{x} \in \Gamma$  the transformed value of  $x \in R$  and  $\tilde{p}_n$  being defined by

$$p_n = e(\tilde{p}_n) = e^{2\pi i \tilde{p}_n}$$

Let further  $x_n$  be the zeros of

$$b(x) := {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi i x\right) .$$

**Lemma:** Putting

$$\tilde{x}_n := x_n - (n-1)$$

it holds

- I)  $b_H(x) := H[b](x) = -ib(x) = A[b'](x)$
- II)  $b'(x_n) \neq 0$  (“separation of zeros” theorem)
- III)  $\tilde{x}_n \in (0, \frac{1}{2})$  .

We note that  $e_n(x) := e(nx)$  builds an orthogonal system of  $L_2^{\#}(0,1)$ .  
With respect to the zeros of the Zeta function we recall

**Lemma:** Let  $N(T)$  the number of zeros of  $\zeta(s)$  in the region  $0 < \sigma < 1$ ,  $0 < t \leq T$ . Then for the consecutive ordinates of the non-trivial zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  it holds ([IvA] 1.4):

$$\text{I)} \quad N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

$$\text{II)} \quad \gamma_n \approx \frac{2\pi n}{\log n}$$

$$\text{III)} \quad N(\gamma_n - 1) < n \leq N(\gamma_n + 1) \quad .$$

## Summary

The today's Fourier analysis of Weyl (periodical, trigonometric) sums in a Banach space framework is proposed to be replaced by a discrete wavelet analysis on the circle in a distributional Hilbert space framework based on the (hypergeometric, non-periodical) Kummer function.

Related to an alternative distribution function of the primes

$$-\log x \quad \rightarrow \quad \mu(x) := \log \frac{\pi}{2} \cot\left(\frac{\pi}{2}x\right) \in L_2^\#(\frac{1}{2}, 1) \cap H_{-1/2}^\#(0, 1) \quad ,$$

we propose the functions  $b(x)$  resp.  $b'(x)$  as mother wavelet for a discrete wavelet analysis on the circle in a distributional  $H_{-1/2}^\#(0, 1)$ –Hilbert space, alternatively to the Hardy-Littlewood circle method with  $e(x)$  and power series analysis on the unit disk  $D := \{z \in \mathbb{Z} \mid |z| < 1\}$ . The reason why not be more generous than a Hilbert scale factor  $\alpha = -1/2$  is about the fact, that

$$\delta \in H_{-1/2-\varepsilon}.$$

We note that

$$1 - b'\left(\frac{x - p_n}{(x - x_n)(x - n)}\right) = \begin{cases} 1 & x = p_n \\ 1 & x = x_n, n \quad x_n \in (n - 1/2, n) \\ \neq 1 & \text{else} \end{cases}$$

The discrete wavelet transforms (as weighted function with respect to the  $H_{-1/2}$ –inner product) is proposed as alternative to the Farey resp. the major and minor arcs concept.

The properties of  $\tilde{x}_n$ ,  $\tilde{p}_n$  and  $\tilde{p}_n - \tilde{x}_n$  are proposed to define appropriate translation and dilatation factors for appropriate discrete wavelet transforms. The dilation factor triggers a contradiction of the analyzing (mother) wavelet function. The translation factor means a shift of the argument along the circle. We note that the index  $n$  counts the number of turns through the circle.

For the alternative prime number counting function  $\pi(x) \approx x \log^{-1} x$  we give a first trial

$$\pi(x) \approx \begin{cases} x \log^{-1} \tan\left(\frac{\pi}{2}x\right) & x \neq n \\ x_n \log^{-1} \tan\left(\frac{\pi}{2}x_n\right) & x = n \end{cases}$$

**Remark:** “S(chwartz)-asymptotics” of generalized functions are also called asymptotic by translation. Asymptotic by translation in combination with large scale dilations are also applied in [ViJ] to prove the prime number theorem in a distributional way.

## The Goldbach Conjectures

A still missing proof of an additive prime number problem is the binary Goldbach conjecture, that

*“all even numbers greater equal 4 can be expressed as the sum of two primes”.*

The (“weak”) ternary Goldbach conjecture (“the 3-primes problem) states, that

*“every odd number greater equal 7 is a sum of three prime numbers”.*

Because of

$$2n + 1 = 2(n - 1) + 3$$

the ternary conjecture is a consequence of the binary conjecture.

An equivalent formulation of the binary Goldbach conjecture is given by:

$$\forall n \in \mathbb{N} \exists j \in \mathbb{N} : \quad p := n - j \text{ and } q := n + j \text{ are prime numbers.}$$

The conceptual idea of the below it about a transfer into appropriate distributional Hilbert space frameworks like

$$H_{-1/2} = \left\{ \psi \mid \|\psi\|_{-1/2}^2 = (A\psi, \psi)_0 < \infty \right\}, \quad H_{-1} = \left\{ \psi \mid \|\psi\|_{-1}^2 = (A\psi, A\psi)_0 < \infty \right\}$$

$$H_0 = \left\{ \psi \mid \|\psi\|_0^2 = (H\psi, H\psi)_0 < \infty \right\}$$

$$H_{10} = \left\{ \psi \mid \|\psi\|_{10}^2 = (S\psi, S\psi)_0 < \infty \right\}$$

being applied to singular integral equations if the form

$$\sum_p \frac{1}{\tilde{x} - \tilde{p}} = \oint \frac{\tilde{\pi}(\mathcal{G})}{4 \sin^2 \frac{\tilde{x} - \mathcal{G}}{2}} d\mathcal{G} \quad .$$

## A Distributional Way to the Prime Number Theorem ([ViJ])

“S(chwartz)-asymptotics” of generalized functions are also called asymptotic by translation. Asymptotic by translation in combination with large scale dilations are applied in [ViJ] to prove the prime number theorem in a distributional way:

The Mangoldt function  $\Lambda(n)$  defines the Chebyshev function ( $a > 1$ )

$$\psi(x) := \sum_{p^n < x} \log p = \sum_{n < x} \Lambda(n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left[ -\frac{\zeta'(s)}{\zeta(s)} \right] x^s \frac{ds}{s} \quad , \quad J(x) = \int_0^\infty \frac{d\psi(x)}{\log x}$$

with

$$\psi'(x) = \sum_{n < x} \Lambda(n) \delta(x - n) \in H_{-1/2-\varepsilon}(R)$$

And

$$H\psi'(x) = \sum_{n < x} \Lambda(n) H\delta(x - n) = \sum_{n < x} \frac{\Lambda(n)}{\pi(x - n)} \in H_{-1/2-\varepsilon}(R) \cdot$$

In order to prove the PNT  $\psi(x) \approx x$  the asymptotic behavior of  $\psi'(x)$  is analyzed by studying the asymptotic properties of the distribution

$$v(x) := \sum_{n < x} \frac{\Lambda(n)}{n} \delta(x - \log n) \in H_{-1/2-\varepsilon}(R)$$

$$Hv(x) = \sum_{n < x} \frac{\Lambda(n)}{n\pi(x - \log n)} \in H_{-1/2-\varepsilon}(R)$$

with its Fourier-Laplace transform for  $\text{Im}(z) > 0$

$$(v(t), e^{izt}) = \sum_1^\infty \frac{\Lambda(n)}{n^{1-it}} = -\frac{\zeta'(1-iz)}{\zeta(1-iz)} \cdot$$

This leads to the Fourier transform on the real axis, in the distributional sense,

$$\hat{v}(x) = -\frac{\zeta'(1-ix)}{\zeta(1-ix)} = \sum_p \log p \left( \sum_{n=1}^\infty p^{-n(1-ix)} \right) \cdot$$

With respect to the asymptotic by translation and dilation we recall from [ViJ]

$$\lim_{h \rightarrow \infty} v(x+h) = 1 \quad \text{in } S'(R) \quad , \quad \lim_{\lambda \rightarrow \infty} \psi'(\lambda x) = 1 \quad \cdot$$

It further holds that

$$\hat{v}(x) - \frac{1}{x-i0} = \hat{v}(x) + Y'(-x) \in L_{loc}^1(R)$$

has tempered growth, whereby

$$w(x) := \frac{1}{x-i0} := \lim_{y \rightarrow 0} \frac{1}{x-iy} = P.v. \frac{1}{x} + i\pi\delta \quad \text{with} \quad \hat{w}(x) = 2\pi e^{-2\pi x} Y(x)$$

being the disk boundary value of the analytic function  $1/z$ ,  $\text{Im}(z) > 0$  ([PeB] I 316), and  $Y$  being the Heaviside function with the integral representation

$$Y(x) = \int_0^x \delta(\tau) d\tau = -\lim_{y \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau + iy} e^{-i\tau x}.$$

Recalling the definition of several Pseudo differential operators on the unit circles

$$\begin{aligned} \tilde{A}u(\varphi) &:= \frac{1}{2\pi} \oint \ln \frac{1}{2 \sin \frac{\varphi - \vartheta}{2}} u(\vartheta) d\vartheta \\ \tilde{H}u(\varphi) &:= \frac{1}{\pi} \oint \frac{1}{2} \cot \frac{\varphi - \vartheta}{2} u(\vartheta) d\vartheta \\ \tilde{S}u(\varphi) &:= \frac{1}{\pi} \oint \frac{u(\vartheta)}{4 \sin^2 \frac{\varphi - \vartheta}{2}} d\vartheta \end{aligned}$$

the analogue von Mangoldt and density distribution function of the primes are given by

$$\tilde{\psi}'(x) = \sum_{n < x} \Lambda(n) \delta(\tilde{x} - \tilde{n}) \in H_{-1/2-\varepsilon}(\Gamma) \quad , \quad \tilde{\pi}'(x) = \sum_p \delta(\tilde{x} - \tilde{p}) \in H_{-1/2-\varepsilon}(\Gamma).$$

From the above it follows

$$(H\tilde{\psi}') (x) = \frac{1}{\pi} \sum_{n < x} \frac{\Lambda(n)}{\tilde{x} - n} = (S\tilde{\psi})(x) \quad , \quad (H\tilde{\pi}') (x) = \frac{1}{\pi} \sum_p \frac{1}{\tilde{x} - \tilde{p}} = (S\tilde{\pi})(x).$$

The explicit formula of the latter equation is given by

$$\sum_p \frac{1}{\tilde{x} - \tilde{p}} = \oint \frac{\tilde{\pi}(\vartheta)}{4 \sin^2 \frac{\tilde{x} - \vartheta}{2}} d\vartheta.$$

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## Appendix

**Remark:** An alternative dissection of the unit boundary circle is provided by the proof of a theorem in the context of conjugate function in the unit circle, which is given in [GaD] Chapter II, §1, in the context of the integral equation procedure of Theodorsen and Garrick for mapping circle-like boundaries on the unit circle the Hilbert transform:

**Theorem:** Let  $f(z) = u(z) + iv(z)$  ( $u, v$  real) be regular in the open unit disk  $|z| < 1$  and continuous on the closed unit disk  $|z| \leq 1$ . Then for  $v(e^{i\varphi})$  the following integral representation of  $u(e^{i\varphi})$  is valid (Cauchy integral):

$$v(e^{i\varphi}) = v(0) + \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\vartheta}) \cot \frac{\varphi - \vartheta}{2} d\vartheta \cdot$$

The essential equation to prove this theorem are built on closed curves  $C_n$  ([GaD] Chapter II, §1), mainly on the unit circle, but with a arc within the disk according to  $|z - e^{i\varphi}| =: \varepsilon_n \xrightarrow{n \rightarrow \infty} 0$  fulfilling

$$f(e^{i\varphi}) = -\frac{1}{2\pi i} \oint_{C_n} f(z) \frac{z + e^{i\varphi}}{z - e^{i\varphi}} \frac{dz}{z} + \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\vartheta}) \cot \frac{\varphi - \vartheta}{2} d\vartheta \cdot$$

For the polynomials  $z^n$  on the unit circle this means

$$\frac{1}{\pi} \int_{0 \rightarrow 2\pi} e^{in\vartheta} \frac{1}{2} \cot \frac{\varphi - \vartheta}{2} d\vartheta = \begin{cases} -ie^{in\varphi} & n = 1, 2, 3, \dots \\ 0 & n = 0 \\ ie^{in\varphi} & n = -1, -2, \dots \end{cases} \cdot$$

## The Hardy-Littlewood Circle Method

Hardy-Ramanujan and later Hardy-Littlewood [HaG] developed an analytical method, called *circle method*, which concerns with additive prime number problems. The basic idea of the circle method is an analysis of the generating power series with respect to the prime number series in the unit disk

$$F(z) = \sum_{p \in P} z^p, \quad |z| < 1, \quad z \in C$$

resp.

$$f(z) := \sum_p (\log z) z^p.$$

Then e.g. the analysis of the binary and ternary Goldbach problem is about an analysis of  $F^2(x)$  resp.  $F^3(x)$ . The knowledge of the distribution of the primes on the circle is essential.

For the ternary problem the analysis for the major arcs leads to the estimate ( $A > 0, n$  odd)

$$R(n) := \sum_{p_1 + p_2 + p_3 = n} (\log p_1)(\log p_2)(\log p_3) = \int_M S^3(\alpha) e^{2\pi i n \alpha} d\alpha = S(n) \frac{n^2}{2} + O\left(\frac{n^2}{\log^A n}\right).$$

The proof is based on the (progressions prime number) theorem of Page, Siegel and Walfish ([ScW] VI, theorem 5.1):

$$\pi(x; k) = \frac{1}{\varphi(k)} \int_2^x \frac{dt}{\log t} + O(x e^{-c \log^d x}) \quad \text{for } x \rightarrow \infty \text{ uniformly for all } k \leq \log^A x$$

and an estimate for the “exponential sum”

$$S(\alpha) := \sum_{p \leq n} e(\alpha p) := \sum_{p \leq n} e^{2\pi i \alpha p}.$$

**Lemma:** For real  $\alpha$  let  $\|\alpha\| := \text{Min}\{\alpha - [\alpha]; [\alpha] + 1 - \alpha\}$ . Then for integers  $M, N$  it holds ([ScW] VII, lemma 2.2)

$$\sum_{M < n \leq N} e(\alpha p) \leq \text{Min}\{N - M; (2\|\alpha\|)^{-1}\}.$$

From [Vil1] X, lemma 1 and notes, we recall the

**Page theorem:** Let  $\pi(N; q, l)$  denote the number of primes  $p$  satisfying  $p \leq N$  and  $p \equiv l \pmod{q}$ , where  $(l, q) = 1$ . Let  $\varepsilon_0, c, c_1$  be fixed positive numbers, and suppose that  $0 < q \leq r^{c_1}$ .

Then, if  $q$  is not exceptional, we have

$$\pi(N; q, l) = \frac{1}{\varphi(q)} \int_2^N \frac{dx}{\log x} + O\left(\frac{Nr^{-\varepsilon}}{r\varphi(q)}\right).$$

In case  $\alpha$  is near to a rational number  $a/q$  with  $q \leq r^{c_1}$  for any fixed  $c_1$ , the **Siegel theorem** with the estimate

$$\pi(N; q, l) = \frac{1}{\varphi(q)} \int_2^N \frac{dx}{\log x} + O(Ne^{(\log N)^{1/2}})$$

enables a stronger approximation of

$$\sum e(\alpha p)$$

without exceptional values of  $q$ .

The key idea of the circle method is to approximate  $r \rightarrow 1$  (e.g. by putting  $r = e^{-1/N}$ ) and to split the interval  $[0, 1]$  into “arcs”, i.e.  $\mathcal{I}$ -intervals. Major arcs are  $\mathcal{I}$ -intervals with a rational number as its center  $a/q$  with “small”  $q$ : for  $\alpha = a/q + \beta$  near a fraction  $a/q$  the number  $\beta$  needs to be small so that there are no other fractions with denominators  $\leq q$  within a distance of  $|\beta|$  of  $a/q$ . This requirement leads to the Farey decomposition of the unit interval. The Farey fractions of order  $N$  are given by

$$F_Q := \left\{ \frac{a}{q} \mid 1 \leq q \leq N, 0 \leq a \leq q, (a, q) = 1 \right\}.$$

The Farey arcs around each of these fractions are defined as follows: let

$\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$  be consecutive fractions in the Farey decomposition of order  $N$ . Then the intervals

$$M_Q(q, a) := \left( \frac{a+a'}{q+q'}, \frac{a+a''}{q+q''} \right) \text{ for } \frac{a}{q} \neq \frac{1}{1}, a \neq 0$$

$$M_Q(1, 1) := \left( 1 - \frac{1}{Q+1}, 0 \right] \cup \left( 0, \frac{1}{Q+1} \right]$$

are disjoint and their union covers the interval  $[0, 1]$ .

In order to derive appropriate estimates for  $R_\nu(N)$  the knowledge of the distribution of the primes is applied to analyze

$$F(r) = \sum_p r^p \cdot$$

In order to prove the 3-prime Goldbach conjecture ( $\nu = 3$ ) for at least  $N \geq N_0 = 3.4 \cdot 10^{43000}$  ([ChZ]) Vinogradov ([Vil]) replaced the infinite power series  $F(z)$  by the finite series (Weyl) exponential sum ( $\alpha \in R$ )

$$S(\alpha) := \sum_{p \leq N} e^{2\pi i p \alpha} \cdot$$

The major arcs were analyzed by using the Siegel-Walfisz prime number theorem

$$\pi(N; q, l) = \frac{1}{\varphi(q)} \int_2^N \frac{dx}{\log x} + O(Ne^{-(\log N)^{1/2}})$$

with

$$\varphi(q) := \sum_{\substack{m \leq q \\ (m, q) = 1}} 1 = q \prod_{p|q} \left(1 - \frac{1}{p}\right) \cdot$$

For the remaining minor arcs related estimate the derived upper bound for  $|S(\alpha)|$  is purely built on number theoretical arguments with respect to trigonometric sums ([Vil1]).

For the 2-prime conjecture Hardy-Littlewood [HaG1] gave the not proven relationship

$$R_2(N) \approx 2 \frac{N}{\log^2 N} \left( \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \right) \cdot \left( \prod_{\substack{p \geq 3 \\ p|N}} \left(1 + \frac{1}{p-2}\right) \right)$$

and therefore

$$R_2(N) > 0 \text{ for } N \geq N_0.$$

Let  $N_r(n) (\cong R_\nu(N))$  denote the number of representations of  $n$  by a sum of primes and

$$v_r(n) := \sum_{p_1 + \dots + p_r = n} \log p_1 \dots \log p_r \cdot$$

then I holds ([HaG] (1.45), (1.47))

$$F^r(z) = \sum_2^\infty N_r(n) z^n \quad \text{resp.} \quad f^r(z) = \sum_2^\infty v_r(n) z^n \cdot$$

Applying the Cauchy integral formula it follows for  $0 < r < 1$ , ([HaG] (5.21))

$$R_r(n) = \frac{1}{2\pi i} \int_{|z|=r} \frac{F^r(z)}{z^n} \frac{dz}{z} = \frac{1}{r^n} = \int_0^1 F^r(re^{2\pi i \theta}) e^{1-2\pi i n \theta} d\theta \cdot$$

The basic idea to prove appropriate approximation estimates was to approximate the boundary of the unit circle by  $r = e^{-1/n}$  applying the Farey dissection of the circle boundary with appropriately defined major and minor arcs.

Assuming that the zeros of all Dirichlet L-functions have real part less than  $\frac{3}{4}$ , Hardy-Littlewood applied the circle method to prove the 3-primes problem for sufficiently large odd numbers ([HaG], [LaE]).

Vinogradov gave a proof w/o the Dirichlet L-functions assumptions, but “only” for sufficiently large  $n$ :

**Theorem** (Vinogradov, 1937 [Vil]), *every sufficiently large odd integer can be written as the sum of three primes (w/o a Dirichlet L-function assumption).*

Vinogradov proved an appropriate estimate of the major and minor arcs in the form

$$\frac{R_3(n)}{\log^3 n} \leq \sum_{\substack{p_1, p_2, p_3 = \text{prim} \\ p_1 + p_2 + p_3 = n}} 1 \approx \frac{n^2}{\log^3 n} \cdot$$

For the major arcs of the binary problem Hardy-Littlewood ([HaG]) showed an appropriate estimate in the form

$$R_2(n) \approx \frac{2n}{\log^2 n} S_2(n) \approx \frac{n}{\log^2 n} \approx \frac{\pi(n)}{\log n}$$

with

$$S_2(n) := \prod_{p|n} \left(1 + \frac{1}{p-1}\right) \prod_{p \nmid n} \left(1 + \frac{1}{(p-1)^2}\right) \cdot$$

An analogue estimate of the minor arcs in the same way as for the ternary problem leads to a *not sufficient estimate* in the form

$$R_2(n) \ll \frac{n}{\log^c n} n^{1/2} \cdot$$

while a behavior same as for the major arcs or better would be required.

With respect to P1 and the  $L_2$  – norm there is an alternative (still not sufficient) estimate in the form

$$(*) \quad \sum_{M=1}^n \left| \int_{\min \text{ or } -\arcs} S^2(\alpha) e^{2m\alpha} d\alpha \right|^2 \ll n \left[ \frac{n}{\log^a n} \right]^2$$

basically due to the Bessel inequality

$$\sum_{m \leq n} |(v, e_m)|^2 \leq \|v\|^2 .$$

### The Hurwitz Zeta function

A function, which is in a sense a generalization of  $\zeta(s)$  is the Hurwitz Zeta function, defined by ([TiE] 2.17:

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \text{for } \operatorname{Re}(s) > 1, \quad 0 < a \leq 1 .$$

For  $a = 1$  resp.  $a = 1/2$  this reduces to

$$\zeta(s) \quad , \quad (2^s - 1)\zeta(s) .$$

There are also other generalized Zeta function, e.g. Lerch or Epstein or Dedekind Zeta function, as well as Zeta functions associated with cusp forms ([IvA] 11.8). Related to the Hurwitz Zeta function is the Dirichlet series ([IvA] 1.8), defined by

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1} \quad \text{for } \operatorname{Re}(s) > 1$$

where for a fixed  $q \geq 0$   $\chi(n)$  is the arithmetical function known as a character modulo  $q$  (for  $q = 1, L(s, \chi) = \zeta(s)$ , if  $(a, q) > 1$  then  $\chi(a) = 0$ ). For  $q = 1$  it holds

$$L(s, \chi_1) = \zeta(s) \prod_p (1 - p^{-s})^{-1} = \prod_p (1 - p^{-s})^{-1} \prod_{p|q} (1 - p^{-s})^{-1} .$$

Thus  $L(s, \chi_1)$  has a first-order pole at  $s = 1$  just like  $\zeta(s)$  and it behaves similar to  $\zeta(s)$  in many other ways, while  $L(s, \chi)$  for  $\chi \neq \chi_1$  is regular for  $\operatorname{Re}(s) > 0$ .

*The Generalized Riemann Hypothesis (GRH) states that all non-trivial zeros of all Dirichlet L-functions have real part equal to  $\frac{1}{2}$ .*

## Notation and Formulas

With respect the notation and reference to the Riemann Hypothesis we refer to [KBr1]:

Let  $H = L_2^*(\Gamma)$  with  $\Gamma = S^1(\mathbb{R}^2)$ , i.e.  $\Gamma$  is the boundary of the unit sphere. Let  $u(s)$  being a  $2\pi$ -periodic function and  $\oint$  denotes the integral from 0 to  $2\pi$  in the Cauchy-sense. Then for  $u \in H := L_2^*(\Gamma)$  with  $\Gamma := S^1(\mathbb{R}^2)$  and for real  $\beta$  Fourier coefficients and norms are defined by

$$u_\nu := \frac{1}{2\pi} \oint u(x) e^{-i\nu x} dx \quad \|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 .$$

Then the Fourier coefficients of the convolution operator

$$(Au)(x) := -\oint \log 2 \sin \frac{x-y}{2} u(y) dy =: \oint k(x-y) u(y) dy$$

are given by

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu .$$

The operator  $A$  enables characterization of the Hilbert spaces  $H_{-1/2}$  and  $H_{-1}$  in the form

$$H_{-1/2} = \left\{ \psi \mid \|\psi\|_{-1/2}^2 = (A\psi, \psi)_0 < \infty \right\}, \quad H_{-1} = \left\{ \psi \mid \|\psi\|_{-1}^2 = (A\psi, A\psi)_0 < \infty \right\} .$$

With respect to the Dirac function we note that building on the Dirichlet kernel there is a formal representation of  $\delta(x)$  in the distribution sense in the form

$$\delta(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} dk = \frac{1}{\pi} \int_0^\pi \cos(kx) dk = \frac{1}{2} \operatorname{sgn}'(x) \in H_{-1/2-\varepsilon}(-\pi, \pi) \subset H_{-1}(-\pi, \pi) .$$

For

$$\sigma := \cot(\pi \circ) = 2 \sum_1^\infty \sin(2\pi m \circ)$$

it holds (see also [ZyA] XIII, (11-3))

$$(A\sigma)(x) = \sum_{-\infty}^{\infty} \frac{1}{2|n|} \sigma_n(x) = \sum_1^\infty \frac{\sin(2\pi nx)}{n} \in H_0^\# .$$

From literature (e.g. [GaD] pp.63, [Grl] 1.441) we recall

$$\frac{1}{2\pi} \oint_{0 \rightarrow 2\pi} e^{in\varphi} \ln \frac{1}{2 \sin \frac{\varphi - \vartheta}{2}} d\vartheta = \begin{cases} -\frac{1}{2n} e^{in\varphi} & n = 1, 2, 3, \dots \\ 0 & n = 0 \\ \frac{1}{2n} e^{in\varphi} & n = -1, -2, \dots \end{cases}$$

$$\frac{1}{\pi} \oint_{0 \rightarrow 2\pi} e^{in\varphi} \frac{1}{2} \cot \frac{\varphi - \vartheta}{2} d\vartheta = \begin{cases} -ie^{in\varphi} & n = 1, 2, 3, \dots \\ 0 & n = 0 \\ ie^{in\varphi} & n = -1, -2, \dots \end{cases}$$

$$\frac{1}{\pi} \oint_{0 \rightarrow 2\pi} e^{in\varphi} \frac{1}{4 \sin^2 \frac{\varphi - \vartheta}{2}} d\vartheta = \begin{cases} -ne^{in\varphi} & n = 1, 2, 3, \dots \\ 0 & n = 0 \\ ne^{in\varphi} & n = -1, -2, \dots \end{cases} .$$

Due to the corresponding property of the Hilbert transform the functions  $\varphi, \varphi_H$  are identical in a weak  $L_2^\#(0,1)$  – sense, i.e. it holds

- i)  $\|\varphi\|_0^2 = \|\varphi_H\|_0^2$
- ii)  $(\varphi, \chi) = (\varphi_H, \chi) \quad \forall \chi \in L_2^\#(0,1)$
- iii)  $(\varphi'_H, \chi)_{-1/2} = (\sigma, \chi)_{-1/2} \quad \forall \chi \in L_2^\#(0,1)$

because of  $(\varphi'_H, \chi)_{-1/2} = (A\varphi'_H, \chi)_0 = (\varphi_H, \chi)_0 = (A\sigma, \chi)_0 = (\sigma, \chi)_{-1/2}$ .

**Remark:** The functions of the Hardy space  $\check{H}(\Gamma)$  of  $L_2$  – functions on the unit disk circle  $\Gamma$  with an analytical continuation inside the unit disk  $D$  can be parametrized by a point of  $z \in D$  by

$$e_z(\varphi) := \frac{1}{ze^{i\varphi} - 1}$$

where the functions  $e_z(\varphi)$  define a linear, continuous mapping according to

$$(f, e_z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\varphi)}{z - e^{i\varphi}} ie^{i\varphi} d\varphi = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\varphi)}{z - y} dy ,$$

which is an isometry of the spaces  $\check{H}(\Gamma)$  and  $\check{H}(D)$ .



**Remark:** The dual space of  $H_{-1/2}^* = H_{1/2} \subset L_2$  is isometric to the classical Hardy space  $\mathbf{H}_2$  of analytical functions in the unit disc with norm

$$\|f(re^{i\varphi})\|_{H_2} := \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\varphi})|^2 d\varphi.$$

It holds

i) If  $f \in \mathbf{H}_2$ , then there exists boundary values  $f(e^{i\varphi}) = \lim_{r \rightarrow 1} f(re^{i\varphi}) \in L_2(-\pi, \pi)$  with

$$\|f\|_{H_2} = \|f(e^{i\varphi})\|_{L_2}$$

ii) If  $f(e^{i\varphi}) = \sum_{-\infty}^{\infty} u_\nu e^{i\nu\varphi} \in H_{1/2}$ , then its Dirichlet extension into the disc is given by ( $z = re^{i\varphi}$ ):

$$F(z) = \sum_{-\infty}^{\infty} u_\nu r^{|\nu|} e^{i\nu\varphi} = \left( \sum_1^{\infty} u_\nu z^\nu \right) + \left( \sum_1^{\infty} u_{-\nu} z^{-\nu} \right)$$

with

$$\|\nabla F\|_0^2 = \sum_{-\infty}^{\infty} |\nu| |u_\nu|^2 = \|f\|_{1/2}^2.$$

**Remark:** The Voronoi summation formula is related to the Dirichlet divisor problem. The Euler function  $\varphi(n)$  is defined as product of all prime divisors of  $n$  ([LaE])

$$\varphi(n) := \sum_{\substack{m \leq n \\ (m,n)=1}} 1 = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

**Remark:** From [SeA] we note that all zeros  $z_n$  of the Kummer function

$${}_1F_1\left(\frac{1}{2}; \frac{3}{2}, z\right) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{z^n}{n!} = \frac{1}{2} \int_0^1 e^{xt} t^{-1/2} dt = z^{-3/4} e^{z/2} M_{\frac{1}{4}, \frac{1}{4}}(z)$$

lie in the half-plane  $\operatorname{Re}(z) > 1/2$  and in the horizontal stripe  $(2n-1)\pi < |\operatorname{Im}(z)| < 2n\pi$ . It holds

$$\int_0^{\infty} x^{s/2} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{s}{2}\right)}{s-1}.$$

**Remark:** With respect to the Kummer function we recall from [LoA] 1.1, [SeA]:

1. The zeros  $z_n$  of the Kummer function  ${}_1F_1(2\pi ix)$  lie in the intervals  $(n-1/2, n)$
2. If  ${}_1F_1 \in L_1(\mathbb{R})$  continuous and differentiable and  $\psi := {}_1F_1' \in L_2(\mathbb{R})$ , then  $\psi$  is a wavelet.

**Remark:** For the expansion of Kummer functions in terms of Laguerre polynomials and Fourier transforms we refer to [PiA]. Putting

$$K(x) := {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -\pi x^2\right) = \frac{1}{x} \int_0^x e^{-\pi t^2} dt \quad \text{and} \quad G(x) := \int_{-\infty}^x e^{-\pi t^2} dt$$

the link between the polynomial function  $z^n$  and the Hermite polynomials  $H_n(z)$  is given by ([CaD]):

$$H_n(x) = \int_{-\infty}^{\infty} \left(z - i \frac{x}{2}\right)^n dG(x) \cdot$$

Alternatively we propose a corresponding polynomial system defined by

$$K_n(x) = \int_{-\infty}^{\infty} \left(z - i \frac{x}{2}\right)^n dK(x) \cdot$$

**Lemma** ([TiE] 3.7): Let  $f(x)$  be positive non-decreasing, and, as  $x \rightarrow \infty$  let

$$\int_1^x f(u) \frac{du}{u} \approx x \cdot$$

then

$$f(x) \approx x \cdot$$

**Examples:**

$$g(x) = \pi(x) \log x \quad \text{and} \quad \int_0^x g(u) d \log u \cdot$$

## Additional formulas

For  $0 \leq r \leq 1$ ,  $\psi \in (0, 2\pi)$ ,  $z = re^{i\psi}$  the function

$$G(z) := G_\psi(r) := \sum_{k=1}^{\infty} \frac{z^k}{k} = \sum_{k=1}^{\infty} \frac{\cos(k\psi)}{k} r^k + i \sum_{k=1}^{\infty} \frac{\sin(k\psi)}{k} r^k .$$

fulfills the properties:

- i)  $G_\psi(0) = 0$
- ii)  $G_\psi(1) = \sum_{k=1}^{\infty} \frac{e^{ik\psi}}{k} = -\log\left(2 \sin \frac{\psi}{2}\right) + i \frac{\pi - \psi}{2}$
- iii)  $G'_\psi(r) = \sum_{k=1}^{\infty} e^{ik\psi} r^{k-1} = \frac{e^{i\psi}}{1 - e^{i\psi r}}$
- iv)  $-\log\left(2 \sin \frac{\psi}{2}\right) = \sum_{k=1}^{\infty} \frac{\cos(k\psi)}{k} = \operatorname{Re}(G(1))$
- v)  $\frac{\pi - \psi}{2} = \sum_{k=1}^{\infty} \frac{\sin(k\psi)}{k} = \operatorname{Im}(G(1))$
- vi)  $G(z) = \log \frac{1}{1-z} = \sum_{k=1}^{\infty} \frac{z^k}{k}$  for  $z \in D$ .

It further holds ([GaD] §3):

$$g(z) := \sum_0^{\infty} a_k z^k \in L_2(D) \quad \Leftrightarrow \quad \|g\|_{L_2(D)} = \pi \sum_0^{\infty} \frac{|a_k|^2}{k+1} < \infty$$

$$P_n(z) := \sum_{k \leq n} a_k z^k \in L_2(D) \quad \Rightarrow \quad \|g - P_n\|_{L_2(D)} = \sum_{k > n} \frac{|a_k|^2}{k+1} \xrightarrow{n \rightarrow \infty} 0 .$$

**Theorems from Polya, Müntz, Ikehara, Wiener, Ramanujan, Nyman, Theodorsen, Frullani, Hardy**

**Theorems from G. Polya**

G. Polya obtained the following general theorem about zeros of the Fourier transform of a real function:

**Theorem 1:** Let  $0 \leq a < b \leq \infty$  and let  $g(x)$  be a strictly positive continuous function on  $(a, b)$  and differentiable there, except possibly at finitely many points. Suppose that

$$\alpha \leq -x \frac{g'(x)}{g(x)} \leq \beta$$

at every point of  $(a, b)$  where  $g(x)$  is differentiable. Suppose further that the integral

$$G(s) := \int_0^{\infty} x^s g(x) \frac{dx}{x}$$

is convergent for  $a^* < \operatorname{Re}(s) < \beta^*$ . Then all zeros  $\rho$  of  $G(s)$  in this stripe satisfy  $\alpha \leq \operatorname{Re}(\rho) \leq \beta$ .

Let

$$g(s) := \frac{\int_1^{\infty} x^{-s} d\mu}{\int_1^{\infty} x^{-s} dx} .$$

**Theorem 2 (G. Polya):** If  $\varphi$  is a polynomial which has all its roots on the imaginary axis, or if  $\varphi$  is an entire function which can be written in a suitable way as limit of such polynomials, then

If (\*)  $\int_0^{\infty} u^{1-s} F(u) \frac{du}{u}$  has all its zeros on the critical line, so does  $\int_0^{\infty} u^{1-s} F(u) \varphi(\log u) \frac{du}{u}$ .

**Modern version:** An operator which takes an even function  $q(v)$  and replaces it by  $\frac{q(v+1) - q(v-1)}{v}$  has the property of moving the zeros of a function closer on the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis.

**Theorem 3 (G. Polya):** If  $\phi_m(x)$  is a polynomial which has all its roots on the imaginary axis, or if it is an entire function which can be written in a suitable way as a limit of such polynomials, then

if  $\int_0^{\infty} x^{-s} F(x) dx$  has all its zeros on the critical axis, so does

$$\int_0^{\infty} x^{-s} F(x) \phi(\log x) dx = \int_{-\infty}^{\infty} e^{-ys} F(e^y) \phi(y) dy .$$

## The Müntz Formula

**Theorem (Müntz' formula):** For  $\omega(x), \omega'(x)$  continuous and bounded in any finite interval with  $\omega(x) = o(x^{-\alpha})$  and  $\omega(x) = o(x^{-\beta})$  for  $x \rightarrow \infty$  and  $\alpha, \beta > 1$  it holds

$$\zeta(s) \int_0^{\infty} x^s \frac{\omega(x) dx}{x} = \int_0^{\infty} x^s \left[ \sum_1^{\infty} \omega(nx) - \frac{1}{x} \int_0^{\infty} \omega(t) dt \right] \frac{dx}{x} \quad \text{for } 0 < \text{Re}(s) < 1.$$

**Proof:** because  $\omega(x)$  is continuous and bounded in any finite interval with  $\omega(x) = o(x^{-\alpha})$  it holds

$$\sum_1^{\infty} \frac{1}{n^s} \left| \int_0^{\infty} x^{s-1} \omega(x) dx \right| \text{ exists for } 1 < \sigma < \alpha,$$

i.e. the inversion leading to the left hand side of (4.3) is justified.

$$\text{ii) } \sum_1^{\infty} \omega(nx) - \int_0^{\infty} \omega(xt) dt = x \int_0^{\infty} \omega'(t)(t - [t]) dt = x \int_0^{1/x} O(1) dt + x \int_{1/x}^{\infty} O((xt)^{-\beta}) dt = O(1)$$

The first summand is justified, because  $\omega(x)$  is continuous and bounded in any finite interval the second summand is justified, because  $\omega(x) = o(x^{-\alpha})$ , i.e. it holds

$$\sum_1^{\infty} \omega(nx) = O(1) + \frac{c}{x} \text{ with } c := \int_0^{\infty} \omega(t) dt.$$

Hence

$$\int_0^{\infty} x^s \sum_1^{\infty} \omega(nx) + \frac{dx}{x} = \int_0^1 x^s \left[ \sum_1^{\infty} \omega(nx) - \frac{c}{x} \right] \frac{dx}{x} + \int_1^{\infty} x^s \sum_1^{\infty} \omega(nx) \frac{dx}{x} + \frac{c}{s-1}$$

for  $\sigma > 0$  except  $s = 1$ . Also

$$-c \int_1^{\infty} x^{s-2} dx = \frac{c}{s-1} \quad \text{for } \sigma < 1$$

and therefore the result for  $0 < \sigma = \text{Re}(s) < 1$  •

## Ikehara's Theorem

If the measure  $d\mu$  is positive and the function  $g(s)$  fulfills

i)  $g(s)$  is properly defined for  $\text{Re}(s) > 0$

ii)  $\lim_{s \rightarrow 1^+} g(s)$  exists for  $s \rightarrow 1^+$  and is written as  $g(1)$

iii)  $\frac{g(s) - g(1)}{(s-1)}$  has a continuous extension from the open halfplane  $\text{Re}(s) > 1$ , (whereby it is necessarily defined and analytical) to the closed halfplane  $\text{Re}(s) \geq 1$ ,

then

$$\text{If } \lim_{s \rightarrow 1^+} \frac{\int_1^\infty x^{-s} d\mu}{\int_1^\infty x^{-s} dx} = 1 \quad \text{for } s \rightarrow 1^+ \quad \text{then} \quad \lim_{A \rightarrow \infty} \frac{\int_1^A d\mu}{\int_1^A dx} = 1 \quad \text{for } A \rightarrow \infty$$

i.e. roughly speaking  $d\mu \approx dx$  in the sense above. The function

$$g(s) := (s-1) \left[ -\frac{\zeta'(s)}{\zeta(s)} \right]$$

gives the prime number theorem. The Siegel formula (see below) might give the link to the Stieltjes density above:

$$g(s) := (s-1)(\zeta(s) \approx 1 + (s-1)/2 + \dots)$$

From [LGa] we recall the two versions of Ikehara theorem:

**Lemma (Ikehara version 1):** Let  $\mu$  be a monotone nondecreasing function on  $(0, \infty)$  and let

$$F(s) = \int_1^\infty x^{1-s} \frac{d\mu(x)}{x} .$$

If the integral converges absolutely for  $\text{Re}(s) > 1$  and there is a constant  $A$  such that

$$F(s) - \frac{A}{s-1}$$

extends to a continuous function in  $\text{Re}(s) \geq 1$  then

$$\mu(x) \approx Ax .$$

**Lemma (Ikehara version 2):** Let the Dirichlets series

$$F(s) = \sum_1^{\infty} \frac{c_n}{n^s}$$

be convergent for  $\text{Re}(s) > 1$ . If there exists a constant  $A$  such that

$$F(s) - \frac{A}{s-1}$$

admits a continuous extension to the line  $\text{Re}(s) \geq 1$ , then

$$\sum_1^N c_n \approx A * N \quad \text{as } N \rightarrow \infty .$$

### Wiener's Tauberian theorem

The closed linear hull of the translates of a function

$$f(x) \in L_1(\mathbb{R})$$

is the whole space  $L_1(\mathbb{R})$  if and only if its Fourier transform

$$\hat{f}(x) := \int_{\mathbb{R}} e^{-ix} f(t) dt$$

never vanishes. Note that the close linear hull in question contains all convolutions

$$f * g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy .$$

## Ramanujan's Master Theorem

**Ramanujan's Master Theorem:** In the neighborhood of  $x = 0$  for

$$F(x) = \sum_0^{\infty} \frac{\varphi(k)}{k!} (-x)^k$$

the following representation holds true

$$\int_0^{\infty} F(x)x^{s-1} dx = \Gamma(s)\varphi(-s) \cdot$$

Ramanujan motivated his formula with the following wordings ([1] B. C. Berndt, chapter 4, Entry 8):

*“Statement: If two functions of  $x$  be equal, then a general theorem can be formed by simply writing  $\varphi(n)$  instead of  $x^n$  in the original theorem*

*Solution: “Put  $x=1$  and multiply it by  $f(0)$  then change  $x$  to  $x, x^2, x^3, x^4 \dots$  and multiply  $\frac{f'(0)}{1!}, \frac{f''(0)}{2!}, \frac{f'''(0)}{3!}, \dots$  respectively and add up all the results. Then instead of  $x^n$  we have  $f(x^n)$  for positive as well as for negative values of  $n$ . Changing  $f(x^n)$  to  $\varphi(n)$  we can get the result.”*

**Example:**

$$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$$

Ramanujan's building process:

$$f(0)[\arctan 1 + \arctan 1] = \frac{\pi}{2} f(0) \cdot$$

$$\frac{f'(0)}{1!} \left[ \arctan x + \arctan \frac{1}{x} \right] = \frac{f'(0)}{1!} \frac{\pi}{2} \cdot$$

$$\frac{f''(0)}{2!} \left[ \arctan x + \arctan \frac{1}{x} \right] = \frac{f''(0)}{2!} \frac{\pi}{2}$$

...



Replace  $\arctan z$  by its Maclaurin series in  $z$ , where  $z$  is any integral power of  $x$ . Now add all the equalities above. On the left side one obtains two double series. Invert the order of summation in each double series to find that

$$\sum_0^{\infty} (-1)^n \frac{f(x^{2n+1}) + f(x^{-2n-1})}{2n+1} = \frac{\pi}{2} f(1) \cdot$$

Replace  $f(x^n)$  by  $\varphi(n)$  to conclude that

$$\sum_0^{\infty} (-1)^n \frac{\varphi(2n+1) + \varphi(-2n-1)}{2n+1} = \frac{\pi}{2} \varphi(0) \cdot$$

Of course, this formal procedure is fraught with numerous difficulties, but the theorem was finally correctly proved by G.H. Hardy.

The link to differential form is given by the Pfaff form  $\omega = -\mu_y(x)dx + \mu_x(y)dy$ .

Let

$$U := \mathbb{R}^2 - \{(0,0)\} \quad \text{and} \quad V := \mathbb{R}^2 - \{(x,0) | x \leq 0\}$$

In case of the (non-star formed) domain  $U$  there is no “integral” for the differential, but this is the case for the domain  $V$ . In this case the “integral” of  $\omega$  is related to one of the “core” functions used by Ramanujan  $\arctan(y(x))$ , which is

$$F(x, y) = \int_{(1,0)}^{(x,y)} \omega = \varphi = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \frac{\pi}{2} - \arctan \frac{y}{x} & y > 0 \\ -\frac{\pi}{2} - \arctan \frac{y}{x} & y < 0 \end{cases}$$

It holds

$$dF(x, y) = 0 \quad , \quad \frac{\partial}{\partial y}(-\mu_y(x)) = \frac{\partial}{\partial x}(-\mu_x(y)) \cdot$$

**Remark** Putting

$$\varphi(k) := \frac{1}{\zeta(2k+1)}$$

the Hardy/Littlewood resp. the Riesz equivalence criteria of the Riemann Hypothesis are

(HL) RH holds if and only if  $F(x) = \sum_0^{\infty} \frac{\varphi(k)}{k!} (-x)^k = O(x^{-1/4}) \cdot$

(R) RH holds if and only if  $\sum_1^{\infty} \frac{(-1)^{k+1}}{(k-1)! \zeta(2k)} x^k = O(x^{1/4+\epsilon}) \cdot$

### Bagchi's Nyman criterion formulation

Let  $H$  denote the weighted  $l^2$  – space consisting of all sequences  $a = \{a_n | n \in \mathbb{N}\}$  of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \omega_n \leq \frac{c_2}{n^2} .$$

Let  $\gamma := \{1, 1, 1, 1, \dots\}$  ,  $\gamma_k := \left\{ \rho\left(\frac{n}{k}\right) \middle| n=1, 2, 3, \dots \right\} \in H$  for  $k = 1, 2, 3, \dots$

and  $\Gamma_k$  be the closed linear span of  $\gamma_k$  . Then the Nyman criterion states

$$\text{The Riemann Hypothesis is true} \quad \Leftrightarrow \quad \gamma \in \bar{\Gamma}_k .$$

### Integral Equation from Theodorsen

For a complex valued function  $2\pi$  – periodic function  $f(\varphi) = u(\varphi) + iv(\varphi)$  its conjugated function can be represented by ([DGA], 1.1, 1.2)

$$\bar{f}(\varphi) = -\lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \oint_{\varepsilon, \pi} f(\varphi + \vartheta) - f(\varphi - \vartheta) \cot \frac{\vartheta}{2} d\vartheta = \frac{1}{2\pi i} \oint_{0, 2\pi} f(\vartheta) \cot \frac{\varphi - \vartheta}{2} d\vartheta .$$

Let  $a_0; a_n, b_n$  be the Fourier coefficients of  $f$  . Then  $0; -b_n, a_n$  are the Fourier coefficients of its conjugate and it holds

$$\frac{1}{\pi} \int_0^{2\pi} f^2(\varphi) d\varphi = \frac{a_0^2}{2} + \frac{1}{\pi} \int_0^{2\pi} \bar{f}^2(\varphi) d\varphi \quad \text{resp.} \quad \frac{1}{\pi} \int_0^{2\pi} \bar{f}^2(\varphi) d\varphi = \sum_1^n a_n^2 + b_n^2 .$$

**Conformal mapping problem:** Let  $C$  be a star-shaped Jordan curve of the  $w$  – plane with respect to  $w = 0$  , let  $\rho = \rho(\theta)$  its representation by polar coordinates. Let  $D$  denote the inner region of  $C$  and  $w = f(z)$  the conformal mapping function with domain  $|z| < 1$  onto  $D$  , which is normalized by  $f(0) = 0$  and  $f'(0) > 0$  . Then the unknown “boundary function”  $\rho = \rho(\vartheta)$  is the solution of the non-linear, singular integral equation from Theodorsen:

$$\theta(\varphi) = \varphi + \frac{1}{2\pi i} \oint_{0, 2\pi} \log \rho(\theta(\vartheta)) \cot \frac{\varphi - \vartheta}{2} d\vartheta .$$

This results in the following representation of the conformal mapping function

$$f(z) = z * \exp \left[ \frac{1}{2\pi i} \oint_{0, 2\pi} \log \rho(\theta(\vartheta)) \frac{e^{i\vartheta} + z}{e^{i\vartheta} - z} d\vartheta \right] \quad |z| < 1 .$$

## Theorem of Frullani

**Lemma (Theorem of Frullani)** Let  $f(x)$  be a continuous, integrable function over any interval  $0 \leq A \leq x \leq B < \infty$ . Then, for  $0 < b < a$ ,

$$\int_0^{\infty} [f(ax) - f(bx)] \frac{dx}{x} = [f(\infty) - F(0)] \log \frac{a}{b}$$

where  $f(0) = \lim_{x \rightarrow 0^+} f(x)$  and  $f(\infty) = \lim_{x \rightarrow \infty} f(x)$ .

We mention a generalization of this lemma, due to Hardy (Quart. J. Math. 33 (1902) p. 113-144) in the form

$$\int_0^{\infty} [\varphi(ax^m) - \psi(bx^n)] (\log x)^p \frac{dx}{x} \cdot$$

## The Hardy Theorem

The Gauss-Weierstrass density function

$$\omega_1(x) := f_\alpha(x) := e^{-\pi\alpha x^2} \quad \text{with } \alpha := 1$$

gives the Jacobi's  $\mathcal{G}$ -relation ([HEd] 1.6ff.)

$$\mathcal{G}(x^2) := G(x) := \sum_{n=-\infty}^{\infty} f(nx) = G(1/x)/x =: 1 + 2\psi(x^2) \cdot$$

A modified integral operator representation ([HEd] 11.1) in the form

$$\frac{2\xi(s)}{s(s-1)} = \int_0^{\infty} x^{1-s} \left[ G(x) - 1 - \frac{1}{x} \right] \frac{dx}{x}$$

is used to prove the **Hardy theorem** ([HEd] 11.1), i.e. that *there are infinitely many roots of  $\xi(s)=0$  on the line  $\text{Re}(s)=1/2$* . If the integral operator would be self-adjoint all zeros have to be on the critical line.

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## Hilbert Scales

There are certain relations between the spaces  $\{H_\alpha \mid \alpha \geq 0\}$  for different indices:

**Lemma:** Let  $\alpha < \beta$ . Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding  $H_\beta \rightarrow H_\alpha$  is compact.

**Lemma:** Let  $\alpha < \beta < \gamma$ . Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with  $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$  and  $\nu = \frac{\beta - \alpha}{\gamma - \alpha}$ .

**Lemma:** Let  $\alpha < \beta < \gamma$ . To any  $x \in H_\beta$  and  $t > 0$  there is a  $y = y_t(x)$  according to

- i)  $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$
- ii)  $\|x - y\|_\beta \leq \|x\|_\beta$ ,  $\|y\|_\beta \leq \|x\|_\beta$
- iii)  $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$ .

**Corollary:** Let  $\alpha < \beta < \gamma$ . To any  $x \in H_\beta$  and  $t > 0$  there is a  $y = y_t(x)$  according to

- i)  $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$  for  $\alpha \leq \rho \leq \beta$
- ii)  $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$  for  $\beta \leq \sigma \leq \gamma$ .

**Remark:** Our construction of the Hilbert scale is based on the operator  $A$  with the two properties i) and ii). The domain  $D(A)$  of  $A$  equipped with the norm

$$\|Ax\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space  $H_2$  which is densely and compactly embedded in  $H = H_0$ . It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator  $A$  with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \quad \text{and} \quad \|x\|_2 = \|Ax\|.$$

For  $t > 0$  we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1}^n e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 .$$

Now the factor have exponential decay  $e^{-\sqrt{\lambda_i} t}$  instead of a polynomial decay in case of  $\lambda_i^\alpha$ . Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t) \|x\|_\alpha \text{ for } x \in H_\alpha$$

with  $c(\alpha, t)$  depending only from  $\alpha$  and  $t > 0$ . Thus the  $(t)$ -norm is weaker than any  $\alpha$ -norm. On the other hand any negative norm, i.e.  $\|x\|_\alpha$  with  $\alpha < 0$ , is bounded by the 0-norm and the newly introduced  $(t)$ -norm. It holds:

**Lemma 5:** Let  $\alpha > 0$  be fixed. The  $\alpha$ -norm of any  $x \in H_0$  is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2$$

with  $\delta > 0$  being arbitrary.

**Remark 2:** This inequality is in a certain sense the counterpart of the logarithmic convexity of the  $\alpha$ -norm, which can be reformulated in the form ( $\mu, \nu > 0, \mu + \nu > 1$ )

$$\|x\|_\beta^2 \leq \nu \varepsilon \|x\|_\gamma^2 + \mu e^{-\nu/\mu} \|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu .$$

The counterpart of lemma 4 above is

**Lemma 6:** Let  $t, \delta > 0$  be fixed. To any  $x \in H_0$  there is a  $y = y_t(x)$  according to

- i)  $\|x - y\| \leq \|x\|$
- ii)  $\|y\|_1 \leq \delta^{-1} \|x\|$
- iii)  $\|x - y\|_{(t)} \leq e^{-t/\delta} \|x\| .$